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A Study on Sum Formulas of Generalized Hexanacci

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• Introduction

The generalized Hexanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r, s, t, u, v, y)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}, \qquad (1.1)$$
$$W_0 = c_0,\ W_1 = c_1,\ W_2 = c_2,\ W_3 = c_3,\ W_4 = c_4,\ W_5 = c_5,\quad n \geq 6$$

where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and r, s, t, u, v, y are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

for $n = 1, 2, 3, \ldots$ when $y \neq 0$. Therefore, recurrence (1.1) holds for all integer n.Hexanacci sequence has been studied by many authors, see for example [1,2,3] and references therein.

Table 1. A few special case of generalized Hexanacci sequences

No	Sequences (Numbers)	Notation	References
1	Generalized Hexanacci	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 1, 1, 1, 1, 1, 1)\}$	[4]
2	Generalized Sixth order Pell	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 2, 1, 1, 1, 1, 1)\}$	[5]
3	Generalized Sixth order Jacobsthal	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 1, 1, 1, 1, 1, 2)\}$	[6]
4	Generalized 6-primes	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 2, 3, 5, 7, 11, 13)\}$	[7]

For some specific values of $W_0, W_1, W_2, W_3, W_4, W_5$ and r, s, t, u, v, y it is worth presenting these special Hexanacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 2) are used for the special cases of r, s, t, u, v, y and initial values.

Sequences (Numbers)	Notation	OEIS [8]
Hexanacci	$\{H_n\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 1)\}$	A001592
Hexanacci-Lucas	$\{E_n\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 1)\}$	A074584
sixth order Pell	$\{P_n^{(6)}\} = \{W_n(0, 1, 2, 5, 13, 34; 2, 1, 1, 1, 1, 1)\}$	
sixth order Pell-Lucas	$\{Q_n^{(6)}\} = \{W_n(6, 2, 6, 17, 46, 122; 2, 1, 1, 1, 1, 1)\}$	
modified sixth order Pell	$\{E_n^{(6)}\} = \{W_n(0, 1, 1, 3, 8, 21; 2, 1, 1, 1, 1, 1)\}$	
sixth order Jacobsthal	$\{J_n^{(6)}\} = \{W_n(0, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 2)\}$	
sixth order Jacobsthal-Lucas	$\{j_n^{(6)}\} = \{W_n(2, 1, 5, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$	
modified sixth order Jacobsthal	$\{K_n^{(6)}\} = \{W_n(3, 1, 3, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$	
sixth-order Jacobsthal Perrin	$\{Q_n^{(6)}\} = \{W_n(3, 0, 2, 8, 16, 32; 1, 1, 1, 1, 1, 2)\}$	
adjusted sixth-order Jacobsthal	$\{S_n^{(6)}\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 2)\}$	
modified sixth-order Jacobsthal-Lucas	$\{R_n^{(6)}\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 2)\}$	
6-primes	$\{G_n\} = \{W_n(0, 0, 0, 0, 1, 2; 2, 3, 5, 7, 11, 13)\}$	
Lucas 6-primes	$\{H_n\} = \{W_n(6, 2, 10, 41, 150, 542; 2, 3, 5, 7, 11, 13)\}$	
modified 6-primes	$\{E_n\} = \{W_n(0, 0, 0, 0, 1, 1; 2, 3, 5, 7, 11, 13)\}$	

Table 2. A few members of generalized Hexanacci sequences

For easy writing, from now on, we drop the superscripts from the sequences, for example we write

n

P_n for $P^{(6)}$.

We present some works on summing formulas of the numbers in the following Table 3.

Table 3. A few special study of sum formulas

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[10,11,12],[13,14]
Generalized Fibonacci	[15,16,17,18,19,20,21]
Generalized Tribonacci	[22,23,24]
Generalized Tetranacci	[25,26,27]
Generalized Pentanacci	[28,29]
Generalized Hexanacci	[30,31]

In this work, we investigate summation formulas of generalized Hexanacci numbers.

- **Sum Formulas of Generalized Hexanacci Numbers with Positive Subscripts**

The following theorem presents some summing formulas of generalized Hexanacci numbers with positive subscripts.

Theorem 2.1. *Let x be a real (or complex) number. For $n \geq 0$ we have the following formulas:*

(a) *If* $sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1 \neq 0$ *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{\Theta_1(x)}{\Theta(x)}$$

where

$\Theta_1(x) = x^{n+5}W_{n+5} - (rx - 1)x^{n+4}W_{n+4} - (sx^2 + rx - 1)x^{n+3}W_{n+3} - (sx^2 + tx^3 + rx - 1)x^{n+2}W_{n+2} - (sx^2 + tx^3 + ux^4 + rx - 1)x^{n+1}W_{n+1} + yx^{n+6}W_n - x^5W_5 + x^4(rx - 1)W_4 + x^3(sx^2 + rx - 1)W_3 + x^2(sx^2 + tx^3 + rx - 1)W_2 + x(sx^2 + tx^3 + ux^4 + rx - 1)W_1 + (sx^2 + tx^3 + ux^4 + vx^5 + rx - 1)W_0$

and

$\Theta(x) = sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1.$

(b) *If* $r^2x + 2ux^2 + 2x^3y - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 - x^6y^2 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 2sx^4y - 2ux^5y - 1 \neq 0$ *then*

where

n
$\sum$
x^kW_{2k}

$k=0$

$= \frac{\Theta_2(x)}{\Delta_1}$

( )
$\Theta_2(x) = -ux^2 + x^3y + sx - 1\ x^{n+1}W_{2n+2} + (t + rs + vx + rx^2y + rux)x^{n+2}W_{2n+1} + (u + t^2x - u^2x^2 + v^2x^3 - x^4y^2 + rt + xy + 2tvx^2 - sx^2y - 2ux^3y + rvx - sux)x^{n+2}W_{2n} + (v + ru + tx^2y - svx + tux + rxy)x^{n+2}W_{2n-1} + (y + v^2x^2 - x^3y^2 + rv - ux^2y + tvx - sxy)x^{n+2}W_{2n-2} + y(r + vx^2 + tx)x^{n+2}W_{2n-3} - x^3(r + vx^2 + tx)W_5 + x^2(r^2x + ux^2 + x^3y + sx + rtx^2 + rvx^3 - 1)W_4 - x^3(t + vx - svx^2 + rx^2y + rux - stx)W_3 + x(r^2x + ux^2 + x^3y - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 + rvx^3 - sux^3 + tvx^4 - sx^4y - 1)W_2 - x^3(v - uvx^2 + tx^2y - svx + rxy)W_1 + (r^2x + 2ux^2 + x^3y - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - sx^4y - ux^5y - 1)\ W_0,$

and

$\Delta_1 = r^2x + 2ux^2 + 2x^3y - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 - x^6y^2 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 2sx^4y - 2ux^5y - 1.$

(c) *If* $r^2x + 2ux^2 + 2x^3y - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 - x^6y^2 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 -$
$\sum$
$2sx^4y - 2ux^5y - 1 = 0$ *then*

where

n

x^kW

$k=0$

$2k+1$

$= \frac{\Theta_3(x)}{\Delta_1}$

$\Theta_3(x) = (r + vx^2 + tx)x^{n+1}W_{2n+2} + (s - s^2x + x^2y + t^2x^2 - u^2x^3 + v^2x^4 - x^5y^2 + ux + rvx^2 - 2sux^2 + 2tvx^3 - 2sx^3y - 2ux^4y + rtx)x^{n+1}W_{2n+1} + (t + vx - svx^2 + rx^2y + rux - stx)\ x^{n+1}W_{2n} + (u - u^2x^2 + v^2x^3 - x^4y^2 + xy + tvx^2 - sx^2y - 2ux^3y + rvx - sux)x^{n+1}W_{2n-1} + (v - uvx^2 + tx^2y - svx + rxy)x^{n+1}W_{2n-2} - yx^{n+1}(ux^2 + x^3y + sx - 1)W_{2n-3} + x^2(ux^2 + x^3y + sx - 1)W_5$

$-x\ (t + rs + vx + rx\ y + rux)W_4 + x(r\ x + ux\ +x\ y - s\ x\ + 2sx + rtx\ + rvx\ - sux\ - sx\ y - 1)$
3 2 2

2 3 2 2

2 3 3 4

$W_3 - x^3$
$v + ru + tx^2y - svx + tux + rxy$
$W_2 + (r^2x + 2ux^2 + x^3y - s^2x^2 + t^2x^3 - u^2x^4 +$

$2sx + 2rtx^2 + rvx^3 - 2sux^3 + tvx^4 - sx^4y - ux^5y - 1)W_1 - x^3y(r + vx^2 + tx)W_0.$

Proof.

- Using the recurrence relation

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$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$$

i.e.

we obtain

$$yW_{n-6} = W_n - rW_{n-1} - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5}$$

$yx^{0}W_{0} = x^{0}W_{6} - rx^{0}W_{5} - sx^{0}W_{4} - tx^{0}W_{3} - ux^{0}W_{2} - vx^{0}W_{1}$ $yx^{1}W_{1} = x^{1}W_{7} - rx^{1}W_{6} - sx^{1}W_{5} - tx^{1}W_{4} - ux^{1}W_{3} - vx^{1}W_{2}$ $yx^{2}W_{2} = x^{2}W_{8} - rx^{2}W_{7} - sx^{2}W_{6} - tx^{2}W_{5} - ux^{2}W_{4} - vx^{2}W_{3}$ $yx^{3}W_{3} = x^{3}W_{9} - rx^{3}W_{8} - sx^{3}W_{7} - tx^{3}W_{6} - ux^{3}W_{5} - vx^{3}W_{4}$

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$yx^{n-4}W_{n-4} = x^{n-4}W_{n+2} - rx^{n-4}W_{n+1} - sx^{n-4}W_{n} - tx^{n-4}W_{n-1} - ux^{n-4}W_{n-2} - vx^{n-4}W_{n-3}$ $yx^{n-3}W_{n-3} = x^{n-3}W_{n+3} - rx^{n-3}W_{n+2} - sx^{n-3}W_{n+1} - tx^{n-3}W_{n} - ux^{n-3}W_{n-1} - vx^{n-3}W_{n-2}$ $yx^{n-2}W_{n-2} = x^{n-2}W_{n+4} - rx^{n-2}W_{n+3} - sx^{n-2}W_{n+2} - tx^{n-2}W_{n+1} - ux^{n-2}W_{n} - vx^{n-2}W_{n-1}$ $yx^{n-1}W_{n-1} = x^{n-1}W_{n+5} - rx^{n-1}W_{n+4} - sx^{n-1}W_{n+3} - tx^{n-1}W_{n+2} - ux^{n-1}W_{n+1} - vx^{n-1}W_{n}$

$yx^{n}W_{n} = x^{n}W_{n+6} - rx^{n}W_{n+5} - sx^{n}W_{n+4} - tx^{n}W_{n+3} - ux^{n}W_{n+2} - vx^{n}W_{n+1}$

If we add the equations side by side (and using $W_{n+6} = rW_{n+5} + sW_{n+4} + tW_{n+3} + uW_{n+2} + vW_{n+1} + yW_n$, , we get(a).

Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$$

i.e.

we obtain

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5} - yW_{n-6}$$

$rx^{1}W_{3} = x^{1}W_{4} - sx^{1}W_{2} - tx^{1}W_{1} - ux^{1}W_{0} - vx^{1}W_{-1} - yx^{1}W_{-2}$ $rx^{2}W_{5} = x^{2}W_{6} - sx^{2}W_{4} - tx^{2}W_{3} - ux^{2}W_{2} - vx^{2}W_{1} - yW_{0}$ $rx^{3}W_{7} = x^{3}W_{8} - sx^{3}W_{6} - tx^{3}W_{5} - ux^{3}W_{4} - vx^{3}W_{3} - yx^{3}W_{2}$ $rx^{4}W_{9} = x^{4}W_{10} - sx^{4}W_{8} - tx^{4}W_{7} - ux^{4}W_{6} - vx^{4}W_{5} - yx^{4}W_{4}$

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$rx^{n-1}W_{2n-1} = x^{n-1}W_{2n} - sx^{n-1}W_{2n-2} - tx^{n-1}W_{2n-3} - ux^{n-1}W_{2n-4} - vx^{n-1}W_{2n-5} - yx^{n-1}W_{2n-6}$ $rx^{n}W_{2n+1} = x^{n}W_{2n+2} - sx^{n}W_{2n} - tx^{n}W_{2n-1} - ux^{n}W_{2n-2} - vx^{n}W_{2n-3} - yx^{n}W_{2n-4}$

Now, if we add the above equations side by side, we get

$$r(-x^{0}W_{1} + \sum_{k=0}^{n} x^{k}W_{2k+1}) = (x^{n}W_{2n+2} - x^{0}W_{2} - x^{-1}W_{0} + \sum_{k=0}^{n} x^{k-1}W_{2k}) \quad (2.1)$$

$$-s(-x^{0}W_{0} + \sum_{k=0}^{n} x^{k}W_{2k}) - t(-x^{n+1}W_{2n+1} + \sum_{k=0}^{n} x^{k+1}W_{2k+1})$$

$$-u(-x^{n+1}W_{2n} + \sum_{k=0}^{n} x^{k+1}W_{2k}) - v(-x^{n+2}W_{2n+1} - x^{n+1}W_{2n-1}$$

$$+x^{1}W_{-1} + \sum_{k=0}^{n} x^{k+2}W_{2k+1})$$

$\sum_{k=0}^{n}$

$-y(-x^{n+2}W_{2n} - x^{n+1}W_{2n-2} + x^{1}W_{-2} + x^{k+2}W_{2k})$

Similarly, using the recurrence relation

$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$

i.e.

$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5} - yW_{n-6}$

we write the following obvious equations;

$$rx^{1}W_{2} = x^{1}W_{3} - sx^{1}W_{1} - tx^{1}W_{0} - ux^{1}W_{-1} - vx^{1}W_{-2} - yx^{1}W_{-3} rx^{2}W_{4} = x^{2}W_{5} - sx^{2}W_{3} - tx^{2}W_{2} - ux^{2}W_{1} - vx^{2}W_{0} - yx^{2}W_{-1}\ rx^{3}W_{6} = x^{3}W_{7} - sx^{3}W_{5} - tx^{3}W_{4} - ux^{3}W_{3} - vx^{3}W_{2} - yx^{3}W_{1}$$

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$$rx^{n-1}W_{2n-2} = x^{n-1}W_{2n-1} - sx^{n-1}W_{2n-3} - tx^{n-1}W_{2n-4} - ux^{n-1}W_{2n-5} - vx^{n-1}W_{2n-6} - yx^{n-1}W_{2n-7} rx^{n}W_{2n} = x^{n}W_{2n+1} - sx^{n}W_{2n-1} - tx^{n}W_{2n-2} - ux^{n}W_{2n-3} - vx^{n}W_{2n-4} - yx^{n}W_{2n-5}$$

Now, if we add the above equations side by side, we obtain

$$r(-x^{0}W_{0} + \sum_{k=0}^{n} x^{k}W_{2k}) = (-x^{0}W_{1} + \sum_{k=0}^{n} x^{k}W_{2k+1}) - s(-x^{n+1}W_{2n+1} + \sum_{k=0}^{n} x^{k+1}W_{2k+1}) \quad (2.2)$$

$\sum_{k=0}^{n}$

$-t(-x^{n+1}W_{2n} + \sum_{k=0}^{n} x^{k+1}W_{2k})$

$\sum_{k=0}^{n}$

$-u(-x^{n+2}W_{2n+1} - x^{n+1}W_{2n-1} + x^{1}W_{-1} + x^{k+2}W_{2k+1})$

$\sum$

$-v(-x^{n+2}W_{2n} - x^{n+1}W_{2n-2} + x^{1}W_{-2} + \sum_{k=0}^{n} x^{k+2}W_{2k})$

$-y(-x^{n+3}W_{2n+1} - x^{n+2}W_{2n-1} - x^{n+1}W_{2n-3}$

$\sum_{k=0}^{n}$

$+x^{1}W_{-3} + x^{2}W_{-1} + x^{k+3}W_{2k+1})$

Then, solving the system (2.1)-(2.2), the required result of (b) and (c) follow. Q

• Special Cases

In this section, for the special cases of *x,* we present the closed form solutions (identities) of the

$k=0$
$k=0$
$k=0$

sums $\sum_n x^k W_k$, $\sum_n x^k W_{2k}$ and $\sum_n x^k W_{2k+1}$ for the specific case of sequence $\{W_n\}$.

- **The case $x = 1$**

In this subsection we consider the special case $x = 1$.

The case $x = 1$ of Theorem 2.1 is given in Soykan [31, Theorem 2.1]. For the generalized 6-primes sequence case ($x = 1, r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$), see [7].

We only consider the case $x = 1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (which is not considered in [31]).

Observe that setting $x = 1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (i.e. for the generalized sixth order Jacobsthal sequence case) in Theorem 2.1 (b), (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 3.1. *If $r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ then for $n \geq 0$ we have the following formulas:*

(a)(b)

n $k=0$

$k=0$

- $\sum_n$

$W_k = {}^1 (W_{n+5} - W_{n+3} - 2W_{n+2} - 3W_{n+1} + 2W_n - W_5 + W_3 + 2W_2 + 3W_1 + 4W_0)$. $W_{2k} = {}^1((n+4)W_{2n+2} - 2(n+3)W_{2n+1} + (10+n)W_{2n} - 2(n+3)W_{2n-1} + (n+7)W_{2n-2} -$

6
9

$2(n+3)W_{2n-3} + 4W_5 - 9W_4 + 4W_3 - 6W_2 + 4W_1 - 3W_0)$.

$k=0$
9

(c) $\sum_n W_{2k+1} = {}^1(-(n+2)W_{2n+2} + 2(n+7)W_{2n+1} - (n+2)W_{2n} + (2n+11)W_{2n-1} - (n+2)W_{2n-2} + 2(n+4)W_{2n-3} - 5W_5 + 8W_4 - 2W_3 + 8W_2 + W_1 + 8W_0)$.

Proof.

- We use Theorem 2.1 (a). If we set $x = 1, r = 1, s = 1, t = 1, u = 2$ in Theorem 2.1 (a) we get(a).
- We use Theorem 2.1 (b). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 2.1 (b) then we have

where

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{g_1(x)}{-(4x-1)(x-1)(x+x^2+1)^2}$$

$g_1(x) = -x^{n+1}(2x^3 + x^2 + x - 1)W_{2n+2} + x^{n+2}(2x^2 + 2x + 2)W_{2n+1} - x^{n+2}(4x^4 + 3x^3 + x^2 - 3x - 2)W_{2n} + x^{n+2}(2x^2 + 2x + 2)W_{2n-1} - x^{n+2}(4x^3 + x^2 + x - 3)W_{2n-2} + 2x^{n+2}(x^2 + x + 1)W_{2n-3} - x^3(x^2 + x + 1)W_5 + x^2(3x^3 + 2x^2 + 2x -$

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$1)W_4 - x^3(x^2+x+1)W_3 + x(-x^4+3x^3+2x^2+3x-1)W_2 - x^3(x^2+x+1)W_1 + (-x^5-x^4+3x^3+3x^2+3x-1)W_0$.

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) using

$\sum$

d 2 2

W

$2k$

$=$

dx

$\frac{d}{dx}(-(4x-1)(x-1)(x+x^2+1)^2)$.

.

n $\frac{d}{dx}(g_1(x))$

$k=0$

$x=1$

$= \frac{1}{}((n+4)W_9$

$2n+2$

$-2($

$$= \frac{g_2(x)}{-(4x-1)(x-1)(x+x^2+1)^2}$$

$g_2(x) = x^{n+1}(x^2+x+1)W_{2n+2} + x^{n+1}(-4x^5-3x^4-3x^3+2x^2+x+1)W_{2n+1} + x^{n+1}$
$(x^2+x+1)W_{2n} - x^{n+1}(4x^4+3x^3+2x^2-2x-1)W_{2n-1} + x^{n+1}(x^2+x+1)W_{2n-2} - 2x^{n+1}$

$(2x^3+x^2+x-1)W_{2n-3} + x^2(2x^3+x^2+x-1)W_5 - x^3(2x^2+2x+2)W_4 + x(-2x^4+2x^3+x^2+3x-1)W_3 - x^3(2x^2+2x+2)W_2 + (-2x^5-2x^4+2x^3+3x^2+3x-1)W_1 - 2x^3(x^2+x+1)W_0$.

.

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (c) using

n

$\sum W$ $=$

$\frac{d}{dx}(g_2(x))$

dx

$k=0$

$2k+1$

d 2 2

$\frac{d}{dx}(-(4x-1)(x-1)(x+x^2+1)^2)$.

$x=1$

$-$

$= \quad 1(\quad (n+2)W$

9

$2n+2$

$+2(n+7)W$

$2n+1$

$-(n+2)W_{2n}$

$+(2n+11)W_{2n-1}$

$$-(n+2)W_{2n-2} + 2(n+4)W_{2n-3} - 5W_5 + 8W_4 - 2W_3 + 8W_2 + W_1 + 8W_0).$$

Q

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ in the last Theorem, we have the following Corollary which presents sum formulas of sixth-order Jacobsthal numbers.

(a)

Corollary 3.2. *For $n \geq 0$, sixth order Jacobsthal numbers have the following properties:*

$n \; k=0$

$k=0$

(b) $\sum_n$

$J_k = {}^1(J_{n+5} - J_{n+3} - 2J_{n+2} - 3J_{n+1} + 2J_n + 5).$

6

9

$J_{2k} = {}^1((n+4)J_{2n+2} - 2(n+3)J_{2n+1} + (10+n)J_{2n} - 2(n+3)J_{2n-1} + (n+7)J_{2n-2} -$

$2(n+3)J_{2n-3} - 3).$

$k=0$

9

(c) $\sum_n \quad J_{2k+1} = {}^1(-(n+2)J_{2n+2} + 2(n+7)J_{2n+1} - (n+2)J_{2n} + (2n+11)J_{2n-1} - (n+2)J_{2n-2} +$

$2(n+4)J_{2n-3} + 10).$

From the last Theorem, we have the following Corollary which gives sum formulas of sixth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$).

(a)

Corollary 3.3. *For $n \geq 0$, sixth order Jacobsthal-Lucas numbers have the following properties:*

$n \; k=0$

$k=0$

(b) $\sum_n$

$j_k = {}^1(j_{n+5} - j_{n+3} - 2j_{n+2} - 3j_{n+1} + 2j_n - 9).$

6

9

$j_{2k} = {}^1((n+4)j_{2n+2} - 2(n+3)j_{2n+1} + (10+n)j_{2n} - 2(n+3)j_{2n-1} + (n+7)j_{2n-2} -$

$2(n+3)j_{2n-3} - 12).$

$k=0$

9

(c) $\sum_n \quad j_{2k+1} = {}^1(-(n+2)j_{2n+2} + 2(n+7)j_{2n+1} - (n+2)j_{2n} + (2n+11)j_{2n-1} - (n+2)j_{2n-2} +$

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$2(n+4)j_{2n-3}-3).$

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$ in the last Theorem, we have the following corollary which presents sum formula of modified sixth order Jacobsthal numbers.

Corollary 3.4. *For $n \geq 0$, modified sixth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n} K_k = \frac{1}{6}(K_{n+5} - K_{n+3} - 2K_{n+2} - 3K_{n+1} + 2K_n - 9).$

(b) $\sum_{k=0}^{n} K_{2k} = \frac{1}{9}((n+4)K_{2n+2} - 2(n+3)K_{2n+1} + (10+n)K_{2n} - 2(n+3)K_{2n-1} + (n+7)K_{2n-2} - 2(n+3)K_{2n-3} - 3).$

(c) $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{9}(-(n+2)K_{2n+2} + 2(n+7)K_{2n+1} - (n+2)K_{2n} + (2n+11)K_{2n-1} - (n+2)K_{2n-2} + 2(n+4)K_{2n-3} - 11).$

From the last Theorem, we have the following corollary which gives sum formula of sixth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$).

Corollary 3.5. *For $n \geq 0$, sixth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{6}(Q_{n+5} - Q_{n+3} - 2Q_{n+2} - 3Q_{n+1} + 2Q_n - 8).$

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{9}((n+4)Q_{2n+2} - 2(n+3)Q_{2n+1} + (10+n)Q_{2n} - 2(n+3)Q_{2n-1} + (n+7)Q_{2n-2} - 2(n+3)Q_{2n-3} - 5).$

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{9}(-(n+2)Q_{2n+2} + 2(n+7)Q_{2n+1} - (n+2)Q_{2n} + (2n+11)Q_{2n-1} - (n+2)Q_{2n-2} + 2(n+4)Q_{2n-3} - 8).$

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ in the Theorem, we have the following corollary which presents sum formula of adjusted sixth-order Jacobsthal numbers.

Corollary 3.6. *For $n \geq 0$, adjusted sixth-order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n} S_k = \frac{1}{6}(S_{n+5} - S_{n+3} - 2S_{n+2} - 3S_{n+1} + 2S_n - 1).$

(b) $\sum_{k=0}^{n} S_{2k} = \frac{1}{9}((n+4)S_{2n+2} - 2(n+3)S_{2n+1} + (10+n)S_{2n} - 2(n+3)S_{2n-1} + (n+7)S_{2n-2} - 2(n+3)S_{2n-3} + 2).$

(c) $\sum_{n} S_{2k+1} = {}^{1}(-(n+2)S_{2n+2}+2(n+7)S_{2n+1} -(n+2)S_{2n}+(2n+11)S_{2n-1} -(n+2)S_{2n-2}+$

$2(n+4)S_{2n-3} -3).$

From the last Theorem, we have the following corollary which gives sum formula of modified sixth- order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$).

Corollary 3.7. *For $n \geq 0$, modified sixth-order Jacobsthal-Lucas numbers have the following property:*

(a)

$n\ k=0$

$k=0$

(b) $\sum_{n}$

$R_k = {}^{1}(R_{n+5} - R_{n+3} - 2R_{n+2} - 3R_{n+1} + 2R_n + 9).$

6

9

$R_{2k} = {}^{1}((n+4)R_{2n+2} - 2(n+3)R_{2n+1} + (10+n)R_{2n} - 2(n+3)R_{2n-1} + (n+7)R_{2n-2} -$

$2(n+3)R_{2n-3} - 15).$

$k=0$

9

(c) $\sum_{n} R_{2k+1} = {}^{1}(-(n+2)R_{2n+2}+2(n+7)R_{2n+1} -(n+2)R_{2n}+(2n+11)R_{2n-1} -(n+2)R_{2n-2}+$

$2(n+4)R_{2n-3} + 24).$

- **The case $x = -1$**

(identities) of the sums

of the sequence $\{W_n\}$.

n

$k=0$

$(-1)^k W_k,$

n

$k=0$

$(-1)^k W_{2k}$ and

n

$k=0$

$(-1)^k W_{2k+1}$ for the specific case

In this subsection we consider the special case $x = -1$ and we present the closed form solutions

Taking $r = s = t = u = v = y = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the followingProposition.

Proposition 3.1. *If $r = s = t = u = v = y = 1$ then for $n \geq 0$ we have the following formulas:*

$k=0$

(a) $\sum_{n} (-1)^k W_k = (-1)^n (W_{n+5} - 2W_{n+4} + W_{n+3} - 2W_{n+2} + W_{n+1} - W_n) - W_5 + 2W_4 - W_3 +$

$2W_2 - W_1 + 2W_0.$

$k=0$

5

(b) $\sum_{n} (-1)^k W_{2k} = {}^{1}((-1)^n (2W_{2n+2} - W_{2n+1} - 2W_{2n-1} - 3W_{2n-2} - W_{2n-3}) - W_5 + 3W_4 -$

$4W_2 - W_1 + 3W_0).$

$k=0$

5

(c) $\sum_{n} (-1)^k W_{2k+1} = {}^{1}((-1)^n (W_{2n+2} + 2W_{2n+1} - W_{2n-1} + W_{2n-2} + 2W_{2n-3}) + 2W_5 - W_4 -$

$5W_3 - 2W_2 + 2W_1 - W_0).$

From the above Proposition, we have the following Corollary which gives sum formulas of Hexanacci numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$).

Corollary 3.8. *For $n \geq 0$, Hexanacci numbers have the following properties:*

(a)

n $k=0$

$k=0$

(b) $\sum_n$

$k=0$

$(-1)^k H_k = (-1)^n (H_{n+5} - 2H_{n+4} + H_{n+3} - 2H_{n+2} + H_{n+1} - H_n) - 1.$ $(-1)^k H_{2k} = \frac{1}{5}((-1)^n (2H_{2n+2} - H_{2n+1} - 2H_{2n-1} - 3H_{2n-2} - H_{2n-3}) - 1).$

5

(c) $\sum_{k=0}^{n} (-1)^k H_{2k+1} = \frac{1}{5}((-1)^n (H_{2n+2} + 2H_{2n+1} - H_{2n-1} + H_{2n-2} + 2H_{2n-3}) + 2).$

Taking $W_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$ in the above Proposition, we have the following Corollary which presents sum formulas of Hexanacci-Lucas numbers.

(a)

Corollary 3.9. *For $n \geq 0$, Hexanacci-Lucas numbers have the following properties:*

n $k=0$

$k=0$

(b) $\sum_n$

$k=0$

$(-1)^k E_k = (-1)^n (E_{n+5} - 2E_{n+4} + E_{n+3} - 2E_{n+2} + E_{n+1} - E_n) + 9.$

$(-1)^k E_{2k} = \frac{1}{5}((-1)^n (2E_{2n+2} - E_{2n+1} - 2E_{2n-1} - 3E_{2n-2} - E_{2n-3}) + 19).$

5

(c) $\sum_{k=0}^{n} (-1)^k E_{2k+1} = \frac{1}{5}((-1)^n (E_{2n+2} + 2E_{2n+1} - E_{2n-1} + E_{2n-2} + 2E_{2n-3}) + 2).$

Taking $r = 2, s = t = u = v = y = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the followingProposition.

Proposition 3.2. *If $r = 2, s = t = u = v = y = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} (-1)^k W_k = \frac{1}{2}((-1)^n (W_{n+5} - 3W_{n+4} + 2W_{n+3} - 3W_{n+2} + 2W_{n+1} - W_n) - W_5 + 3W_4 - 2W_3 + 3W_2 - 2W_1 + 3W_0).$

(b) $\sum_{k=0}^{n} (-1)^k W_{2k} = \frac{1}{4}((-1)^n (W_{2n+2} - W_{2n+1} - W_{2n-1} - 2W_{2n-2} - W_{2n-3}) - W_5 + 3W_4 - 3W_2 + 3W_0).$

(c) $\sum_{k=0}^{n} (-1)^k W_{2k+1} = \frac{1}{4}((-1)^n (W_{2n+2} + W_{2n+1} - W_{2n-1} + W_{2n-3}) + W_5 - W_4 - 4W_3 - W_2 + 2W_1 - W_0).$

From the last Proposition, we have the following Corollary which gives sum formulas of sixth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$).

(a)

2

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Corollary 3.10. *For $n \geq 0$, sixth-order Pell numbers have the following properties:*

n $k=0$

$k=0$

(b) $\sum_n$

$k=0$

$(-1)^k P_k = {}^1 ((-1)^n (P_{n+5} - 3P_{n+4} + 2P_{n+3} - 3P_{n+2} + 2P_{n+1} - P_n) - 1)$. $(-1)^k P_{2k} = {}^1 ((-1)^n (P_{2n+2} - P_{2n+1} -$

$P_{2n-1} - 2P_{2n-2} - P_{2n-3}) - 1)$.

4

(c) $\sum_n \quad (-1)^k P_{2k+1} = {}^1((-1)^n (P_{2n+2} + P_{2n+1} - P_{2n-1} + P_{2n-3}) + 1)$.

4

Taking $W_n = Q_n$ with $Q_0 = 6$, $Q_1 = 2$, $Q_2 = 6$, $Q_3 = 17$, $Q_4 = 46$, $Q_5 = 122$ in the last Proposition, we have the following Corollary which presents sum formulas of sixth-order Pell-Lucas numbers.

(a)

2

Corollary 3.11. *For $n \geq 0$, sixth-order Pell-Lucas numbers have the following properties:*

n $k=0$

$k=0$

(b) $\sum_n$

$k=0$

$(-1)^k Q_k = {}^1 ((-1)^n (Q_{n+5} - 3Q_{n+4} + 2Q_{n+3} - 3Q_{n+2} + 2Q_{n+1} - Q_n) + 14)$. $(-1)^k Q_{2k} = {}^1 ((-1)^n (Q_{2n+2} - Q_{2n+1} -$

$Q_{2n-1} - 2Q_{2n-2} - Q_{2n-3}) + 16)$.

4

(c) $\sum_n \quad (-1)^k Q_{2k+1} = {}^1((-1)^n (Q_{2n+2} + Q_{2n+1} - Q_{2n-1} + Q_{2n-3}))$.

4

Observe that setting $x = -1$, $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 1$, $y = 2$ (i.e. for the generalized sixth order Jacobsthal case) in Theorem 2.1 (a), makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 3.12. *If $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 2$, $y = 2$ then for $n \geq 0$ we have the following formulas:*

$2(-1)$

$(n+3)W_{n+2} - (-1)$

$(n-1)W_{n+1} + 2(-1)$

$(n+6)W_n + 5W_5 - 9W_4 + 2W_3 - 6W_2 -$

$k=0$

n

9

n

n

(a) $\sum_n \quad (-1)^k W_k = {}^1(-(-1)^n (n+5)W_{n+5} + (-1)^n (2n+9)W_{n+4} - (-1)^n (n+2)W_{n+3} +$

$W_1 - 3W_0)$.

$k=0$

10

(b) $\sum_n \quad (-1)^k W_{2k} = \frac{1}{} ((-1)^n (3W_{2n+2} - 2W_{2n+1} + 3W_{2n} - 2W_{2n-1} - 7W_{2n-2} - 2W_{2n-3}) - W_5 +$

$4W_4 - W_3 - 6W_2 - W_1 + 4W_0)$.

k=0
10

(c) $\sum_{k=0}^{n} (-1)^k W_{2k+1} = \frac{1}{10}((-1)^n (W_{2n+2}+6W_{2n+1}+W_{2n} -4W_{2n-1}+W_{2n-2}+6W_{2n-3})+3W_5 -$

$2W_4 - 7W_3 - 2W_2 + 3W_1 - 2W_0)$.

Proof.

- We use Theorem 2.1 (a). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 2.1 (a) then we have

where

$$\sum_{k=0}^{n} x^k W_k = \frac{g_3(x)}{(2x-1)(x+1)(-x+x^2+1)(x+x^2+1)}$$

$g_3(x) = x^{n+5}W_{n+5} - x^{n+4}(x-1)W_{n+4} - x^{n+3}(x^2+x-1)W_{n+3} - x^{n+2}(x^3+x^2+x-1)W_{n+2} - x^{n+1}(x^4+x^3+x^2+x-1)W_{n+1} + 2x^{n+6}W_n - x^5W_5 + x^4(x-1)W_4 + x^3(x^2+x-1)W_3 + x^2(x^3+x^2+x-1)W_2 + x(x^4+x^3+x^2+x-1)W_1 + (x^5+x^4+x^3+x^2+x-1)W_0$.

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) using

$$\sum_{k=0}^{n}(-1)^k W_k = \frac{\frac{d}{dx}(g_3(x))}{\frac{d}{dx}((2x-1)(x+1)(-x+x^2+1)(x+x^2+1))}\Bigg|_{x=-1}$$

$$= \frac{1}{9}(-(-1)^n(n+5)W_{n+5} + (-1)^n(2n+9)W_{n+4} - (-1)^n(n+2)W_{n+3} + 2(-1)^n(n+3)W_{n+2} - (-1)^n(n-1)W_{n+1} + 2(-1)^n(n+6)W_n + 5W_5 - 9W_4 + 2W_3 - 6W_2 - W_1 - 3W_0).$$

(b) Take $x = -1, r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ in Theorem 2.1 (b).

(c) Take $x = -1, r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ in Theorem 2.1 (b). Q

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Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ in the last Theorem, we have the following Corollary which presents sum formulas of sixth-order Jacobsthal numbers.

Corollary 3.13. *For $n \geq 0$, sixth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^kJ_k = \frac{1}{9}(-(-1)^n(n+5)J_{n+5}+(-1)^n(2n+9)J_{n+4}-(-1)^n(n+2)J_{n+3}+2(-1)^n(n+3)J_{n+2}-(-1)^n(n-1)J_{n+1}+2(-1)^n(n+6)J_n-9).$

(b) $\sum_{k=0}^{n}(-1)^kJ_{2k} = \frac{1}{10}((-1)^n(3J_{2n+2}-2J_{2n+1}+3J_{2n}-2J_{2n-1}-7J_{2n-2}-2J_{2n-3})-5).$

(c) $\sum_{k=0}^{n}(-1)^kJ_{2k+1} = \frac{1}{10}((-1)^n(J_{2n+2}+6J_{2n+1}+J_{2n}-4J_{2n-1}+J_{2n-2}+6J_{2n-3})-5).$

From the last Theorem, we have the following Corollary which gives sum formulas of sixth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$).

Corollary 3.14. *For $n \geq 0$, sixth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^kj_k = \frac{1}{9}(-(-1)^n(n+5)j_{n+5}+(-1)^n(2n+9)j_{n+4}-(-1)^n(n+2)j_{n+3}+2(-1)^n(n+3)j_{n+2}-(-1)^n(n-1)j_{n+1}+2(-1)^n(n+6)j_n+3).$

(b) $\sum_{k=0}^{n}(-1)^kj_{2k} = \frac{1}{10}((-1)^n(3j_{2n+2}-2j_{2n+1}+3j_{2n}-2j_{2n-1}-7j_{2n-2}-2j_{2n-3})+7).$

(c) $\sum_{k=0}^{n}(-1)^kj_{2k+1} = \frac{1}{10}((-1)^n(j_{2n+2}+6j_{2n+1}+j_{2n}-4j_{2n-1}+j_{2n-2}+6j_{2n-3})-1).$

Taking $j_n = W_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$ in the last Theorem, we have the following corollary which presents sum formula of modified sixth order Jacobsthal numbers.

Corollary 3.15. *For $n \geq 0$,modified sixth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n}(-1)^kK_k = \frac{1}{9}(-(-1)^n(n+5)K_{n+5}+(-1)^n(2n+9)K_{n+4}-(-1)^n(n+2)K_{n+3}+2(-1)^n(n+3)K_{n+2}-(-1)^n(n-1)K_{n+1}+2(-1)^n(n+6)K_n+12).$

(b) $\sum_{k=0}^{n}(-1)^kK_{2k} = \frac{1}{10}((-1)^n(3K_{2n+2}-2K_{2n+1}+3K_{2n}-2K_{2n-1}-7K_{2n-2}-2K_{2n-3})+23).$

(c) $\sum_{k=0}^{n}(-1)^kK_{2k+1} = \frac{1}{10}((-1)^n(K_{2n+2}+6K_{2n+1}+K_{2n}-4K_{2n-1}+K_{2n-2}+6K_{2n-3})+1).$

From the last Theorem, we have the following corollary which gives sum formula of sixth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$).

Corollary 3.16. *For $n \geq 0$, sixth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=0}^{n}(-1)^kQ_k = \frac{1}{9}(-(-1)^n(n+5)Q_{n+5}+(-1)^n(2n+9)Q_{n+4}-(-1)^n(n+2)Q_{n+3}+2(-1)^n(n+3)Q_{n+2}-(-1)^n(n-1)Q_{n+1}+2(-1)^n(n+6)Q_n+11).$

(b) $\sum_{k=0}^{n}(-1)^kQ_{2k} = \frac{1}{10}((-1)^n(3Q_{2n+2}-2Q_{2n+1}+3Q_{2n}-2Q_{2n-1}-7Q_{2n-2}-2Q_{2n-3})+24).$

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(c) $\sum_{k=0}^{n} (-1)^k Q_{2k+1} = \frac{1}{} ((-1)^n (Q_{2n+2} + 6Q_{2n+1} + Q_{2n} - 4Q_{2n-1} + Q_{2n-2} + 6Q_{2n-3}) - 2).$

Taking $W_n = S_n$ with $S_0 = 0$, $S_1 = 1$, $S_2 = 1$, $S_3 = 2$, $S_4 = 4$, $S_5 = 8$ in the Theorem, we have the following corollary which presents sum formula of adjusted sixth-order Jacobsthal numbers.

Corollary 3.17. *For $n \geq 0$, adjusted sixth-order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n} (-1)^k S_k = \frac{1}{9}(-(-1)^n (n+5)S_{n+5} + (-1)^n (2n+9)S_{n+4} - (-1)^n (n+2)S_{n+3} + 2(-1)^n (n+3)S_{n+2} - (-1)^n (n-1)S_{n+1} + 2(-1)^n (n+6)S_n + 1).$

(b) $\sum_{k=0}^{n} (-1)^k S_{2k} = \frac{1}{10} ((-1)^n (3S_{2n+2} - 2S_{2n+1} + 3S_{2n} - 2S_{2n-1} - 7S_{2n-2} - 2S_{2n-3}) - 1).$

(c) $\sum_{k=0}^{n} (-1)^k S_{2k+1} = \frac{1}{10} ((-1)^n (S_{2n+2} + 6S_{2n+1} + S_{2n} - 4S_{2n-1} + S_{2n-2} + 6S_{2n-3}) + 3).$

From the last Theorem, we have the following corollary which gives sum formula of modified sixth- order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 6$, $R_1 = 1$, $R_2 = 3$, $R_3 = 7$, $R_4 = 15$, $R_5 = 31$).

Corollary 3.18. *For $n \geq 0$, modified sixth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=0}^{n} (-1)^k R_k = \frac{1}{9}(-(-1)^n (n+5)R_{n+5} + (-1)^n (2n+9)R_{n+4} - (-1)^n (n+2)R_{n+3} + 2(-1)^n (n+3)R_{n+2} - (-1)^n (n-1)R_{n+1} + 2(-1)^n (n+6)R_n - 3).$

(b) $\sum_{k=0}^{n} (-1)^k R_{2k} = \frac{1}{10} ((-1)^n (3R_{2n+2} - 2R_{2n+1} + 3R_{2n} - 2R_{2n-1} - 7R_{2n-2} - 2R_{2n-3}) + 27).$

(c) $\sum_{k=0}^{n} (-1)^k R_{2k+1} = \frac{1}{10} ((-1)^n (R_{2n+2} + 6R_{2n+1} + R_{2n} - 4R_{2n-1} + R_{2n-2} + 6R_{2n-3}) - 1).$

Taking $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ in Theorem 2.1 (a), (b) and (c), we obtain thefollowing proposition.

Proposition 3.3. *If $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ then for $n \geq 0$ we have the followingformulas:*

(a) $\sum_{k=0}^{n} (-1)^k W_k = \frac{1}{4}((-1)^n (2W_{n+1} + 5W_{n+2} + 3W_{n+4} - W_{n+5} + 13W_n) + W_5 - 3W_4 - 5W_2 - 2W_1 - 9W_0).$

(b) $\sum_{k=0}^{n} (-1)^k W_{2k} = \frac{1}{82} ((-1)^n (5W_{2n+2} - 6W_{2n+1} + 54W_{2n} - 31W_{2n-1} - 109W_{2n-2} - 52W_{2n-3}) - 4W_5 + 13W_4 + 6W_3 - 8W_2 - 3W_1 + 17W_0).$

(c) $\sum_{k=0}^{n} (-1)^k W_{2k+1} = \frac{1}{82} ((-1)^n (4W_{2n+2} + 69W_{2n+1} - 6W_{2n} - 74W_{2n-1} + 3W_{2n-2} + 65W_{2n-3}) + 5W_5 - 6W_4 - 28W_3 - 31W_2 - 27W_1 - 52W_0).$

From the last proposition, we have the following corollary which gives sum formulas of 6-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$).

(a)

Corollary 3.19. *For $n \geq 0$, 6-primes numbers have the following properties:*

- $\sum_{k=0}^{n}$

k=0

$(-1)^k G_k = {}^1((-1)^n (2G_{n+1} + 5G_{n+2} + 3G_{n+4} - G_{n+5} + 13G_n) - 1).$

4

82

$(-1)^k G_{2k} = \frac{1}{} ((-1)^n (5G_{2n+2} - 6G_{2n+1} + 54G_{2n} - 31G_{2n-1} - 109G_{2n-2} - 52G_{2n-3}) + 5).$

82

(c) $\sum_n \quad (-1)^k G_{2k+1} = \frac{1}{} ((-1)^n (4G_{2n+2} + 69G_{2n+1} - 6G_{2n} - 74G_{2n-1} + 3G_{2n-2} + 65G_{2n-3}) + 4).$

Taking $W_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 6-primes numbers.

Σa)

Corollary 3.20. *For $n \geq 0$, Lucas 6-primes numbers have the following properties:*

$n \; k=0$

$k=0$

(b) $\sum_n$

$k=0$

$(-1)^k H_k = {}^1((-1)^n (2H_{n+1} + 5H_{n+2} + 3H_{n+4} - H_{n+5} + 13H_n) - 16).$

4

82

$(-1)^k H_{2k} = \frac{1}{} ((-1)^n (5H_{2n+2} - 6H_{2n+1} + 54H_{2n} - 31H_{2n-1} - 109H_{2n-2} - 52H_{2n-3}) + 44).$

82

(c) $\sum_n \quad (-1)^k H_{2k+1} = \frac{1}{} ((-1)^n (4H_{2n+2} + 69H_{2n+1} - 6H_{2n} - 74H_{2n-1} + 3H_{2n-2} + 65H_{2n-3}$

$) - 14).$

From the last proposition, we have the following corollary which gives sum formulas of modified 6-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$).

Σa)

Corollary 3.21. *For $n \geq 0$, modified 6-primes numbers have the following properties:*

$n \; k=0$

$k=0$

(b) $\sum_n$

$k=0$

$(-1)^k E_k = {}^1((-1)^n (2E_{n+1} + 5E_{n+2} + 3E_{n+4} - E_{n+5} + 13E_n) - 2).$

4

82

$(-1)^k E_{2k} = \frac{1}{} ((-1)^n (5E_{2n+2} - 6E_{2n+1} + 54E_{2n} - 31E_{2n-1} - 109E_{2n-2} - 52E_{2n-3}) + 9).$

82

(c) $\sum_n \quad (-1)^k E_{2k+1} = \frac{1}{} ((-1)^n (4E_{2n+2} + 69E_{2n+1} - 6E_{2n} - 74E_{2n-1} + 3E_{2n-2} + 65E_{2n-3}) - 1).$

- **The case $x = i$**

In this subsection we consider the special case $x = i$.

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Taking $x = i, r = s = t = u = v = y = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the followingproposition.

Proposition 3.4. *If $r = s = t = u = v = y = 1$ then for $n \geq 0$ we have the following formulas:*

$k=0$
$-2+i$

(a) $\sum_n \quad i^k W_k = \frac{1}{\quad} (i^n(iW_{n+5}+(1-i)W_{n+4}-(1+2i)W_{n+3}-2W_{n+2}+iW_{n+1}-W_n)-iW_5-(1-i)$
$W_4 + (1+2i)W_3 + 2W_2 - iW_1 - (1-i)W_0)$.

$k=0$
$4+i$

(b) $\sum_n \quad i^k W_{2k} = \frac{1}{\quad} (i^n(-2iW_{2n+2}+(1+2i)W_{2n+1}+(1+3i)W_{2n}+(1+i)W_{2n-1}+(2+i)W_{2n-2}+$
$iW_{2n-3}) + W_5 - 3W_4 + (1-i)W_3 + (1+3i)W_2 - iW_1 + (4-i)W_0)$.

$k=0$
$4+i$

(c) $\sum_n \quad i^k W_{2k+1} = \frac{1}{\quad} (i^n(W_{2n+2} + (1+i)W_{2n+1} + (1-i)W_{2n} + (2-i)W_{2n-1} - iW_{2n-2} -$
$2iW_{2n-3}) - 2W_5 + (2-i)W_4 + (2+3i)W_3 + (1-i)W_2 + (5-i)W_1 + W_0)$.

From the above Proposition, we have the following Corollary which gives sum formulas of Hexanacci numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$).

Corollary 3.22. *For $n \geq 0$, Hexanacci numbers have the following properties:*

(a)

$n\ k=0$

$k=0$

(b) $\sum_n$

$i^k H_k = \frac{1}{\quad} (i^n(iH_{n+5} + (1-i)H_{n+4} - (1+2i)H_{n+3} - 2H_{n+2} + iH_{n+1} - H_n) - i)$. $i^k H_{2k} = \frac{1}{\quad} (i^n(-2iH_{2n+2} + (1+2i)H_{2n+1} + (1+3i)H_{2n}$
$+(1+i)H_{2n-1} + (2+i)H_{2n-2} +$

$4+i$
$-2+i$

(c)

$iH_{2n-3}) - 1)$.

$4+i$

$n\ k=0$
2).

$i^k H_{2k+1} = \frac{1}{\quad} (i^n(H_{2n+2}+(1+i)H_{2n+1}+(1-i)H_{2n}+(2-i)H_{2n-1}-iH_{2n-2}-2iH_{2n-3})+$

Taking $H_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$ in the above Proposition, we have the following Corollary which presents sum formulas of Hexanacci-Lucas numbers.

(a)

Corollary 3.23. *For $n \geq 0$, Hexanacci-Lucas numbers have the following properties:*

$n\ k=0$

$k=0$

(b) $\sum_n$

$i^k E_k = \frac{1}{\quad} (i^n(iE_{n+5}+(1-i)E_{n+4} - (1+2i)E_{n+3} - 2E_{n+2} + iE_{n+1} - E_n) + (-8+3i))$. $i^k E_{2k} = \frac{1}{\quad} (i^n(-2iE_{2n+2}+(1+2i)E_{2n+1}+(1+3i)E_{2n}+(1$
$+i)E_{2n-1}+(2+i)E_{2n-2}+$

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$4+i$
$-2+i$

$iE_{2n-3}) + (20 - 5i))$.

$k=0$
$4+i$

(c) $\sum_{n} i^k E_{2k+1} = \frac{1}{} (i^n(E_{2n+2}+(1+i)E_{2n+1}+(1-i)E_{2n}+(2-i)E_{2n-1}-iE_{2n-2}-2iE_{2n-3})+$

$(-4+2i))$.

Corresponding sums of the other sixth order generalized Hexanacci numbers can be calculatedsimilarly.

• Sum Formulas of Generalized Hexanacci Numbers with Negative Subscripts

The following Theorem presents some summing formulas of generalized Hexanacci numbers with negative subscripts.

Theorem 4.1. *Let x be a real (or complex) number. For* $n \geq 1$ *we have the following formulas:*

(a) *If* $y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6 \neq 0$, *then*

n

$\sum x^k W_{-k}$

$$= \frac{\Theta_4(x)}{y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6}$$

where

$k=1$

$\Theta_4(x) = -x^{n+1}W_{-n+5} + (r-x)x^{n+1}W_{-n+4} + (s+rx-x^2)x^{n+1}W_{-n+3} + (t+rx^2+sx-x^3)x^{n+1}W_{-n+2} + (u+rx^3+sx^2+tx-x^4)x^{n+1}W_{-n+1} + (v+rx^4+sx^3+tx^2+ux-x^5)x^{n+1}W_{-n} + xW_5 - x(r-x)W_4 + x(-s-rx+x^2)W_3 + x(-t-rx^2-sx+x^3)W_2 + x(-u-rx^3-sx^2-tx+x^4)W_1 + x(-v-rx^4-sx^3-tx^2-ux+x^5)W_0.$

(b) *If* $-2sx^5 - 2ux^4 - v^2x - 2x^3y - r^2x^5 + s^2x^4 - t^2x^3 + u^2x^2 + y^2 + x^6 - 2rtx^4 - 2rvx^3 + 2sux^3 - 2tvx^2 + 2sx^2y + 2uxy \neq 0$ *then*

where

n

$\sum x^k W_{-2k}$

$k=1$

$$= \frac{\Theta_5(x)}{\Delta_2}$$

$\Theta_5(x) = (-y - sx^2 - ux + x^3)x^{n+1}W_{-2n+4} + x^{n+1}(tx^2 + ry + vx + rsx^2 + rux)W_{-2n+3} + (-2sx^3 - ux^2 - r^2x^3 + s^2x^2 + sy - xy + x^4 - rtx^2 - rvx + sux)x^{n+1}W_{-2n+2} + (vx^2 + ty + rux^2 - svx + tux + rxy)x^{n+1}W_{-2n+1} + (-2sx^4 + u^2x - 2ux^3 - x^2y - r^2x^4 + s^2x^3 - t^2x^2 + uy + x^5 - 2rtx^3 - rvx^2 + 2sux^2 - tvx + sxy)x^{n+1}W_{-2n} + y(v + rx^2 + tx)x^{n+1}W_{-2n-1} - x(v + rx^2 + tx)W_5 + x(y + sx^2 + r^2x^2 + rv + ux - x^3 + rtx)W_4 - x(tx^2 - sv + ry + vx + rux - stx)W_3 + x(2sx^3 + t^2x +$

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$ux^2 + r^2x^3 - s^2x^2 + tv - sy + xy - x^4 + 2rtx^2 + rvx - sux)W_2 - x(vx^2 - uv + ty - svx + rxy)W_1 + x(2sx^4 - u^2x + 2ux^3 + x^2y + r^2x^4 - s^2x^3 + t^2x^2 - uy + v^2 - x^5 + 2rtx^3 + 2rvx^2 - 2sux^2 + 2tvx - sxy)\, W_0,$

$\Delta_2 = -2sx^5 - 2ux^4 - v^2x - 2x^3y - r^2x^5 + s^2x^4 - t^2x^3 + u^2x^2 + y^2 + x^6 - 2rtx^4 - 2rvx^3 + 2sux^3 - 2tvx^2 + 2sx^2y + 2uxy.$

(c) *If* $-2sx^5 - 2ux^4 - v^2x - 2x^3y - r^2x^5 + s^2x^4 - t^2x^3 + u^2x^2 + y^2 + x^6 - 2rtx^4 - 2rvx^3 + 2sux^3 - 2tvx^2 + 2sx^2y + 2uxy \neq 0$ *then*

$$\sum_{k=1}^{n} x^k W_{-2k+1} = \frac{\Theta_6(x)}{\Delta_2}$$

where

$\Theta_6(x) = (v + rx^2 + tx)x^{n+2}W_{-2n+4} + (-y - sx^2 - r^2x^2 - rv - ux + x^3 - rtx)x^{n+2}W_{-2n+3} + (tx^2 - sv + ry + vx + rux - stx)x^{n+2}W_{-2n+2} + (-2sx^3 - t^2x - ux^2 - r^2x^3 + s^2x^2 - tv + sy - xy + x^4 - 2rtx^2 - rvx + sux)x^{n+2}W_{-2n+1} + (vx^2 - uv + ty - svx + rxy)x^{n+2}W_{-2n} + y(-y - sx^2 - ux + x^3)\, x^{n+1}W_{-2n-1} + x(y + sx^2 + ux - x^3)W_5 - x(tx^2 + ry + vx + rsx^2 + rux)W_4 + x(2sx^3 + ux^2 + r^2x^3 - s^2x^2 - sy + xy - x^4 + rtx^2 + rvx - sux)W_3 - x(vx^2 + ty + rux^2 - svx + tux + rxy)W_2 + x(2sx^4 - u^2x + 2ux^3 + x^2y + r^2x^4 - s^2x^3 + t^2x^2 - uy - x^5 + 2rtx^3 + rvx^2 - 2sux^2 + tvx - sxy)\, W_1 - xy(v + rx^2 + tx)W_0.$

Proof.

- Using the recurrence relation

$$W_{-n} = \frac{1}{y}W_{-n+6} - \frac{r}{y}W_{-n+5} - \frac{s}{y}W_{-n+4} - \frac{t}{y}W_{-n+3} - \frac{u}{y}W_{-n+2} - \frac{v}{y}W_{-n+1}$$

i.e.

$$yW_{-n} = W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - vW_{-n+1}$$

we obtain

$$yx^nW_{-n} = x^nW_{-n+6} - rx^nW_{-n+5} - sx^nW_{-n+4} - tx^nW_{-n+3} - ux^nW_{-n+2} - vx^nW_{-n+1}$$

$$yx^{n-1}W_{-n+1} = x^{n-1}W_{-n+7} - rx^{n-1}W_{-n+6} - sx^{n-1}W_{-n+5} - tx^{n-1}W_{-n+4} - ux^{n-1}W_{-n+3} - vx^{n-1}W_{-n+2}$$

$$yx^{n-2}W_{-n+2} = x^{n-2}W_{-n+8} - rx^{n-2}W_{-n+7} - sx^{n-2}W_{-n+6} - tx^{n-2}W_{-n+5} - ux^{n-2}W_{-n+4} - vx^{n-2}W_{-n+3}$$

$$\vdots$$

$$yx^3W_{-3} = x^3W_3 - rx^3W_2 - sx^3W_1 - tx^3W_0 - ux^3W_{-1} - vx^3W_{-2}$$

$$yx^2W_{-2} = x^2W_4 - rx^2W_3 - sx^2W_2 - tx^2W_1 - ux^2W_0 - vx^2W_{-1}$$

$$yx^1W_{-1} = x^1W_5 - rx^1W_4 - sx^1W_3 - tx^1W_2 - ux^1W_1 - vx^1W_0.$$

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If we add the above equations side by side, we get

$\sum$

$y(\sum_{k=1}^{n} x^{k}W_{-k}) = (-x^{n+1}W_{-n+5} - x^{n+2}W_{-n+4} - x^{n+3}W_{-n+3} - x^{n+4}W_{-n+2} - x^{n+5}W_{-n+1}$

$\sum$

$-x^{n+6}W_{-n} + x^{1}W_{5} + x^{2}W_{4} + x^{3}W_{3} + x^{4}W_{2} + x^{5}W_{1} + x^{6}W_{0} + \sum_{k=1}^{n} x^{k+6}W_{-k})$

$-r(-x$

W_{-n}

$+x^{1}W_{4} + x^{2}W_{3} + x^{3}W_{2} + x^{4}W_{1} + x^{5}W_{0} + \sum_{k=1} x^{k+5}W_{-k})$

$-s(-x$

W_{-n}

$+x^{1}W_{3} + x^{2}W_{2} + x^{3}W_{1} + x^{4}W_{0} + \sum_{k=1} x^{k+4}W_{-k})$

$\sum_{k=1}^{n}$

$-t(-x^{n+1}W_{-n+2} - x^{n+2}W_{-n+1} - x^{n+3}W_{-n} + x^{1}W_{2} + x^{2}W_{1} + x^{3}W_{0} + x^{k+3}W_{-k})$

$\sum$

$-u(-x^{n+1}W_{-n+1} - x^{n+2}W_{-n} + x^{1}W_{1} + x^{2}W_{0} + x^{k+2}W_{-k})$ $\sum_{k=1}^{n}$

$\sum_{k=1}^{n}$

$-v(-x^{n+1}W_{-n} + x^{1}W_{0} + x^{k+1}W_{-k})$

and then the desired result follows.

- **and (c)** Using the recurrence relation

$$W_{-n+6} = rW_{-n+5} + sW_{-n+4} + tW_{-n+3} + uW_{-n+2} + vW_{-n+1} + yW_{-n}$$

i.e.

we obtain

$$vW_{-n+1} = W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - yW_{-n}$$

$vx^{n}W_{-2n+1} = x^{n}W_{-2n+6} - rx^{n}W_{-2n+5} - sx^{n}W_{-2n+4} - tx^{n}W_{-2n+3} - ux^{n}W_{-2n+2} - yx^{n}W_{-2n}$
$vx^{n-1}W_{-2n+3} = x^{n-1}W_{-2n+8} - rx^{n-1}W_{-2n+7} - sx^{n-1}W_{-2n+6} - tx^{n-1}W_{-2n+5}$
$-ux^{n-1}W_{-2n+4} - yx^{n-1}W_{-2n+2}$

$vx^{n-2}W_{-2n+5} = x^{n-2}W_{-2n+10} - rx^{n-2}W_{-2n+9} - sx^{n-2}W_{-2n+8} - tx^{n-2}W_{-2n+7}$
$-ux^{n-2}W_{-2n+6} - yx^{n-2}W_{-2n+4}$

.
.
.

$vx^{3}W_{-5} = x^{3}W_{0} - rx^{3}W_{-1} - sx^{3}W_{-2} - tx^{3}W_{-3} - ux^{3}W_{-4} - yx^{3}W_{-6}$
$vx^{2}W_{-3} = x^{2}W_{2} - rx^{2}W_{1} - sx^{2}W_{0} - tx^{2}W_{-1} - ux^{2}W_{-2} - yx^{2}W_{-4}$
$vx^{1}W_{-1} = x^{1}W_{4} - rx^{1}W_{3} - sx^{1}W_{2} - tx^{1}W_{1} - ux^{1}W_{0} - yx^{1}W_{-2}$

If we add the above equations side by side, we get

$$v\sum_{k=1}^{n} x^{k}W_{-2k+1} \quad (4.1)$$

$= (-x^{n+1}W_{-2n+4} - x^{n+2}W_{-2n+2} - x^{n+3}W_{-2n} + x^{3}W_{0} + x^{2}W_{2} + x^{1}W_{4} + \sum_{k=1}^{n} x^{k+3}W_{-2k})$

$-r(-x^{n+1}W_{-2n+3} - x^{n+2}W_{-2n+1} + x^{1}W_{3} + x^{2}W_{1} + \sum_{k=1}^{n} x^{k+2}W_{-2k+1})$

$-s(-x^{n+1}W_{-2n+2} - x^{n+2}W_{-2n} + x^{2}W_{0} + x^{1}W_{2} + \sum_{k=1}^{n} x^{k+2}W_{-2k})$

$$-t(-x^{n+1}W_{-2n+1} + x^{1}W_{1} + \sum_{k=1}^{n} x^{k+1}W_{-2k+1})$$

$$-u(-x^{n+1}W_{-2n} + x^{1}W_{0} + \sum_{k=1}^{n} x^{k+1}W_{-2k}) - y(\sum_{k=1}^{n} x^{k}W_{-2k}).$$

Similarly, using the recurrence relation

$$W_{-n+6} = rW_{-n+5} + sW_{-n+4} + tW_{-n+3} + uW_{-n+2} + vW_{-n+1} + yW_{-n}$$

i.e.

$$vW_{-n+1} = W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - yW_{-n}$$

we obtain

$$vx^{n}W_{-2n} = x^{n}W_{-2n+5} - rx^{n}W_{-2n+4} - sx^{n}W_{-2n+3} - tx^{n}W_{-2n+2} - ux^{n}W_{-2n+1} - yx^{n}W_{-2n-1}$$

$$vx^{n-1}W_{-2n+2} = x^{n-1}W_{-2n+7} - rx^{n-1}W_{-2n+6} - sx^{n-1}W_{-2n+5} - tx^{n-1}W_{-2n+4} - ux^{n-1}W_{-2n+3} - yx^{n-1}W_{-2n+1}$$

$$vx^{n-2}W_{-2n+4} = x^{n-2}W_{-2n+9} - rx^{n-2}W_{-2n+8} - sx^{n-2}W_{-2n+7} - tx^{n-2}W_{-2n+6} - ux^{n-2}W_{-2n+5} - yx^{n-2}W_{-2n+3}$$

$$\vdots$$

$$vx^{3}W_{-6} = x^{3}W_{-1} - rx^{3}W_{-2} - sx^{3}W_{-3} - tx^{3}W_{-4} - ux^{3}W_{-5} - yx^{3}W_{-7}$$

$$vx^{2}W_{-4} = x^{2}W_{1} - rx^{2}W_{0} - sx^{2}W_{-1} - tx^{2}W_{-2} - ux^{2}W_{-3} - yx^{2}W_{-5}$$

$$vx^{1}W_{-2} = x^{1}W_{3} - rx^{1}W_{2} - sx^{1}W_{1} - tx^{1}W_{0} - ux^{1}W_{-1} - yx^{1}W_{-3}$$

If we add the equations side by side, we get

$$v\sum_{k=1}^{n} x^{k}W_{-2k} = (-x^{n+1}W_{-2n+3} - x^{n+2}W_{-2n+1} + x^{2}W_{1} + x^{1}W_{3} + \sum_{k=1}^{n} x^{k+2}W_{-2k+1}) \quad (4.2)$$

$$-r(-x^{n+1}W_{-2n+2} - x^{n+2}W_{-2n} + x^{1}W_{2} + x^{2}W_{0} + \sum_{k=1}^{n} x^{k+2}W_{-2k})$$

$$-s(-x^{n+1}W_{-2n+1} + x^{1}W_{1} + \sum_{k=1}^{n} x^{k+1}W_{-2k+1})$$

$$-t(-x^{n+1}W_{-2n} + x^{1}W_{0} + \sum_{k=1}^{n} x^{k+1}W_{-2k}) - u(\sum_{k=1}^{n} x^{k}W_{-2k+1})$$

$$-y(x^{n}W_{-2n-1} - x^{0}W_{-1} + \sum_{k=1}^{n} x^{k-1}W_{-2k+1}).$$

- **Specific Cases**

In this section, for the specific cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=1}^{n} x^{k}W_{-k}$, $\sum_{k=1}^{n} x^{k}W_{-2k}$ and $\sum_{k=1}^{n} x^{k}W_{-2k+1}$ for the specific case of sequence $\{W_n\}$.

- **The case $x = 1$**

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In this subsection we consider the special case $x = 1$.

The case $x = 1$ of Theorem 2.1 is given in Soykan [31, Theorem 3.1]. For the generalized 6-primes sequence case ($x = 1, r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$), see [7].

We only consider the case $x = 1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (which is not considered in [31]).

Observe that setting $x = 1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (i.e. for the generalized sixth order Jacobsthal sequence case) in Theorem 4.1 (b), (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

6

Theorem 5.1. *If $r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ then for $n \geq 1$ we have the following formulas:*

k=1

(a) $\sum_n$

$W_{-k} = {}^1 (-W_{-n+5}+W_{-n+3}+2W_{-n+2}+3W_{-n+1}+4W_{-n}+W_5-W_3-2W_2-3W_1-4W_0)$.

k=1
9

(b) $\sum_n$ $W_{-2k} = {}^1((n+1)W_{-2n+4} - 2(n+2)W_{-2n+3} + (n+4)W_{-2n+2} - 2(n+2)W_{-2n+1} + (7+ n)W_{-2n} - 2(n+2)W_{-2n-1} + 2W_5 - 3W_4 + 2W_3 - 6W_2 + 2W_1 - 9W_0)$.

k=1
9

(c) $\sum_n$ $W_{-2k+1} = {}^1(-(n+3)W_{-2n+4} + (2n+5)W_{-2n+3} - (n+3)W_{-2n+2} + 2(n+4)W_{-2n+1} - (n+3)W_{-2n} + 2(n+1)W_{-2n-1} - W_5 + 4W_4 - 4W_3 + 4W_2 - 7W_1 + 4W_0)$.

Proof.

- We use Theorem 4.1 (a). If we set $x = 1, r = 1, s = 1, t = 1, u = 2$ in Theorem 4.1 (a) we get(a).
- We use Theorem 4.1 (b). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 4.1 (b) then we have

where

$$\sum_{k=1}^{n} x^k W_{-2k} = \frac{g_4(x)}{(x-1)(x-4)(x+x^2+1)^2}$$

$g_4(x) = -x(x + x^2 + 1)(-x^n(x - 2)W_{-2n+4} - 2x^nW_{-2n+3} - x^n(-4x + x^2 + 2)W_{-2n+2} - 2x^nW_{-2n+1} - x^n(-4x^2 + x^3 + 2)W_{-2n} - 2x^nW_{-2n-1} + W_5 + (x - 3)W_4 + W_3 + (x^2 + 1 - 4x)W_2 + W_1 + (1 - 4x^2 + x^3)W_0)$.

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

dx

$\sum$

dx

.

n — $\frac{d}{}(g_4(x))$

$k=1$

W_{-2k} =

$_d$ $((x-1)(x-4)(x+x_2+1)^2)$.

$x=1$

= $\underline{1}((n+1)W9$

$-$
$2n+4$

$-2(n+2)W_{-2n+3}$

$+(n+4)W_-$

$2n+2$

$-2(n+2)W_-$

$2n+1$

$+(7+n)W_{-2n}-2(n+2)W_{-2n-1}+2W_5-3W_4+2W_3-6W_2+2W_1-9W_0).$

- We use Theorem 4.1 (c). If we set $r=1, s=1, t=1, u=2$ in Theorem 4.1 (c) then we have

where

n

$\sum$

$x^k W_-$

$k=1$

$2k+1$

$$= \frac{g_5(x)}{(x-1)(x-4)(x+x_2+1)^2}$$

$g_5(x) = -x(x+x^2+1)(-x^{n+1}W_{-2n+4} - x^{n+1}(x-3)W_{-2n+3} - x^{n+1}W_{-2n+2} - x^{n+1}(-4x+x^2+1)W_{-2n+1} - x^{n+1}W_{-2n} - 2x^n(x-2)W_{-2n-1} + (x-2)W_5 + 2W_4 + (x^2+2-4x)W_3 + 2W_2 + (2-4x^2+x^3)W_1 + 2W_0)$.

For $x=1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (c) by using

dx
n
dx

$\sum$ $\underline{d}(g_5(x))$.

$k=1$

W_{-2k+1} =

d $((x-1)(x-4)(x+x_2+1)^2)$.

$x=1$

9

$=$ $1(-(n+3)W_-$

$2n+4$

$+(2n+5)W_{-2n+3}$

$-(n+3)W_-$

$2n+2$

$+2(n+4)W_-$

$2n+1$

$-(n+3)W_{-2n}+2(n+1)W_{-2n-1}-W_5+4W_4-4W_3+4W_2-7W_1+4W_0).$

Q

Taking $W_n=J_n$ with $J_0=0, J_1=1, J_2=1, J_3=1, J_4=1, J_5=1$ in the last Proposition, we have the following Corollary which presents sum formulas of sixth-order Jacobsthal numbers.

Corollary 5.2. *For $n\geq 1$, sixth order Jacobsthal numbers have the following properties:*

(a)

(b)

n $k=1$ n $k=1$

$J_{-k}=\frac{1}{6}(-J_{-n+5}+J_{-n+3}+2J_{-n+2}+3J_{-n+1}+4J_{-n}-5).$

9

$J_{-2k}=\frac{1}{9}((n+1)J_{-2n+4}-2(n+2)J_{-2n+3}+(n+4)J_{-2n+2}-2(n+2)J_{-2n+1}+(7+$

$n)J_{-2n}-2(n+2)J_{-2n-1}-3).$

$k=1$

9

(c) $\sum_{k=1}^{n} J_{-2k+1}=\frac{1}{9}(-(n+3)J_{-2n+4}+(2n+5)J_{-2n+3}-(n+3)J_{-2n+2}+2(n+4)J_{-2n+1}-(n+$

$3)J_{-2n}+2(n+1)J_{-2n-1}-4).$

From the last Theorem, we have the following Corollary which gives sum formulas of sixth order Jacobsthal-Lucas numbers (take $W_n=j_n$ with $j_0=2, j_1=1, j_2=5, j_3=10, j_4=20, j_5=40$).

(a)

Corollary 5.3. *For $n\geq 1$, sixth order Jacobsthal-Lucas numbers have the following properties:*

(b)

n $k=1$ n $k=1$

$j_{-k}=\frac{1}{6}(-j_{-n+5}+j_{-n+3}+2j_{-n+2}+3j_{-n+1}+4j_{-n}+9).$

$$j_{-2k} = \frac{1}{9}((n+1)j_{-2n+4} - 2(n+2)j_{-2n+3} + (n+4)j_{-2n+2} - 2(n+2)j_{-2n+1} + (7+n)j_{-2n} - 2(n+2)j_{-2n-1} - 6).$$

(c) $\sum_{k=1}^{n} j_{-2k+1} = \frac{1}{9}(-(n+3)j_{-2n+4} + (2n+5)j_{-2n+3} - (n+3)j_{-2n+2} + 2(n+4)j_{-2n+1} - (n+3)j_{-2n} + 2(n+1)j_{-2n-1} + 21).$

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$ in the last Theorem, we have the following corollary which presents sum formula of modified sixth order Jacobsthal numbers.

Corollary 5.4. *For $n \geq 1$, modified sixth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n} K_{-k} = \frac{1}{6}(-K_{-n+5} + K_{-n+3} + 2K_{-n+2} + 3K_{-n+1} + 4K_{-n} + 9).$

(b) $\sum_{k=1}^{n} K_{-2k} = \frac{1}{9}((n+1)K_{-2n+4} - 2(n+2)K_{-2n+3} + (n+4)K_{-2n+2} - 2(n+2)K_{-2n+1} + (7+n)K_{-2n} - 2(n+2)K_{-2n-1} - 3).$

(c) $\sum_{k=1}^{n} K_{-2k+1} = \frac{1}{9}(-(n+3)K_{-2n+4} + (2n+5)K_{-2n+3} - (n+3)K_{-2n+2} + 2(n+4)K_{-2n+1} - (n+3)K_{-2n} + 2(n+1)K_{-2n-1} + 17).$

From the last Theorem, we have the following corollary which gives sum formula of sixth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$).

Corollary 5.5. *For $n \geq 1$, sixth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{6}(-Q_{-n+5} + Q_{-n+3} + 2Q_{-n+2} + 3Q_{-n+1} + 4Q_{-n} + 8).$

(b) $\sum_{k=1}^{n} Q_{-2k} = \frac{1}{9}((n+1)Q_{-2n+4} - 2(n+2)Q_{-2n+3} + (n+4)Q_{-2n+2} - 2(n+2)Q_{-2n+1} + (7+n)Q_{-2n} - 2(n+2)Q_{-2n-1} - 7).$

(c) $\sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{9}(-(n+3)Q_{-2n+4} + (2n+5)Q_{-2n+3} - (n+3)Q_{-2n+2} + 2(n+4)Q_{-2n+1} - (n+3)Q_{-2n} + 2(n+1)Q_{-2n-1} + 20).$

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ in the last Theorem, we have the following corollary which presents sum formula of adjusted sixth-order Jacobsthal numbers.

Corollary 5.6. *For $n \geq 1$, adjusted sixth-order Jacobsthal numbers have the following property:*

$S_{-k} = \frac{1}{6}(-S_{-n+5} + S_{-n+3} + 2S_{-n+2} + 3S_{-n+1} + 4S_{-n} + 1).$

9

$S_{-2k} = \frac{1}{}((n+1)S_{-2n+4} - 2(n+2)S_{-2n+3} + (n+4)S_{-2n+2} - 2(n+2)S_{-2n+1} + (7+$

$n)S_{-2n} - 2(n+2)S_{-2n-1} + 4).$

k=1

9

(c) $\sum_{k=1}^{n} S_{-2k+1} = \frac{1}{9}(-(n+3)S_{-2n+4} + (2n+5)S_{-2n+3} - (n+3)S_{-2n+2} + 2(n+4)S_{-2n+1} - (n+$

$3)S_{-2n} + 2(n+1)S_{-2n-1} - 3).$

From the last Theorem, we have the following corollary which gives sum formula of modified sixth- order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$).

Corollary 5.7. *For $n \geq 1$, modified sixth-order Jacobsthal-Lucas numbers have the followingproperty:*

(a)

n k=1

k=1

(b) $\sum_n$

$R_{-k} = \frac{1}{6}(-R_{-n+5} + R_{-n+3} + 2R_{-n+2} + 3R_{-n+1} + 4R_{-n} - 9).$

9

$R_{-2k} = \frac{1}{}((n+1)R_{-2n+4} - 2(n+2)R_{-2n+3} + (n+4)R_{-2n+2} - 2(n+2)R_{-2n+1} + (7+$

$n)R_{-2n} - 2(n+2)R_{-2n-1} - 39).$

k=1

9

(c) $\sum_{k=1}^{n} R_{-2k+1} = \frac{1}{9}(-(n+3)R_{-2n+4} + (2n+5)R_{-2n+3} - (n+3)R_{-2n+2} + 2(n+4)R_{-2n+1} -$

$(n+3)R_{-2n} + 2(n+1)R_{-2n-1} + 30).$

- **The case $x = -1$**

In this subsection we consider the special case $x = -1$.

Taking $r = s = t = u = v = y = 1$ in Theorem 4.1 (a) and (b) (or (c)), we obtain the followingProposition.

Proposition 5.1. *If $r = s = t = u = v = y = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} (-1)^k W_{-k} = (-1)^n(-W_{-n+5} + 2W_{-n+4} - W_{-n+3} + 2W_{-n+2} - W_{-n+1} + 2W_{-n}) + W_5 -$

$2W_4 + W_3 - 2W_2 + W_1 - 2W_0.$

(b) $\sum_{k=1}^{n} (-1)^k W_{-2k} = \frac{1}{5}((-1)^n(2W_{-2n+4} - W_{-2n+3} - 5W_{-2n+2} - 2W_{-2n+1} + 2W_{-2n} - W_{-2n-1}) +$

$W_5 - 3W_4 + 4W_2 + W_1 - 3W_0).$

(c) $\sum_{k=1}^{n} (-1)^k W_{-2k+1} = \frac{1}{5}((-1)^n(W_{-2n+4} - 3W_{-2n+3} + 4W_{-2n+1} + W_{-2n} + 2W_{-2n-1}) - 2W_5 +$

$W_4 + 5W_3 + 2W_2 - 2W_1 + W_0).$

From the above Proposition, we have the following Corollary which gives sum formulas of Hexanacci numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$).

Corollary 5.8. *For $n \geq 1$, Hexanacci numbers have the following properties:*

k=1

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(a) $\sum_{k=1}^{n}(-1)^k H_{-k} = (-1)^n(-H_{-n+5} + 2H_{-n+4} - H_{-n+3} + 2H_{-n+2} - H_{-n+1} + 2H_{-n}) + H_5 - 2H_4 + H_3 - 2H_2 + H_1 - 2H_0.$

(b) $\sum_{k=1}^{n}(-1)^k H_{-2k} = \frac{1}{5}((-1)^n(2H_{-2n+4} - H_{-2n+3} - 5H_{-2n+2} - 2H_{-2n+1} + 2H_{-2n} - H_{-2n-1}) + H_5 - 3H_4 + 4H_2 + H_1 - 3H_0).$

(c) $\sum_{k=1}^{n}(-1)^k H_{-2k+1} = \frac{1}{5}((-1)^n(H_{-2n+4} - 3H_{-2n+3} + 4H_{-2n+1} + H_{-2n} + 2H_{-2n-1}) - 2H_5 + H_4 + 5H_3 + 2H_2 - 2H_1 + H_0).$

Taking $W_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$ in the above Proposition, we have the following Corollary which presents sum formulas of Hexanacci-Lucas numbers.

Corollary 5.9. *For $n \geq 1$, Hexanacci-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k E_{-k} = (-1)^n(-E_{-n+5} + 2E_{-n+4} - E_{-n+3} + 2E_{-n+2} - E_{-n+1} + 2E_{-n}) + E_5 - 2E_4 + E_3 - 2E_2 + E_1 - 2E_0.$

(b) $\sum_{k=1}^{n}(-1)^k E_{-2k} = \frac{1}{5}((-1)^n(2E_{-2n+4} - E_{-2n+3} - 5E_{-2n+2} - 2E_{-2n+1} + 2E_{-2n} - E_{-2n-1}) + E_5 - 3E_4 + 4E_2 + E_1 - 3E_0).$

(c) $\sum_{k=1}^{n}(-1)^k E_{-2k+1} = \frac{1}{5}((-1)^n(E_{-2n+4} - 3E_{-2n+3} + 4E_{-2n+1} + E_{-2n} + 2E_{-2n-1}) - 2E_5 + E_4 + 5E_3 + 2E_2 - 2E_1 + E_0).$

Taking $r = 2, s = t = u = v = y = 1$ in Theorem 4.1 (a), (b) and (c), we obtain the followingProposition.

Proposition 5.2. *If $r = 2, s = t = u = v = y = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}(-1)^k W_{-k} = \frac{1}{2}((-1)^n(-W_{-n+5} + 3W_{-n+4} - 2W_{-n+3} + 3W_{-n+2} - 2W_{-n+1} + 3W_{-n}) + W_5 - 3W_4 + 2W_3 - 3W_2 + 2W_1 - 3W_0).$

(b) $\sum_{k=1}^{n}(-1)^k W_{-2k} = \frac{1}{4}((-1)^n(W_{-2n+4} - W_{-2n+3} - 4W_{-2n+2} - W_{-2n+1} + 2W_{-2n} - W_{-2n-1}) + W_5 - 3W_4 + 3W_2 - 3W_0).$

(c) $\sum_{k=1}^{n}(-1)^k W_{-2k+1} = \frac{1}{4}((-1)^n(W_{-2n+4} - 3W_{-2n+3} + 3W_{-2n+1} + W_{-2n-1}) - W_5 + W_4 + 4W_3 + W_2 - 2W_1 + W_0).$

From the last Proposition, we have the following Corollary which gives sum formulas of sixth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$).

(a)

2

Corollary 5.10. *For $n \geq 1$, sixth-order Pell numbers have the following properties:*

n $k=1$

$k=1$

(b) $\sum_n$

$k=1$

$(-1)^k P_{-k} = {}^1 ((-1)^n (-P_{-n+5} + 3P_{-n+4} - 2P_{-n+3} + 3P_{-n+2} - 2P_{-n+1} + 3P_{-n}) + 1)$. $(-1)^k P_{-2k} = {}^1((-1)^n (P_{-2n+4} - P_{-2n+3} - 4P_{-2n+2} - P_{-2n+1} + 2P_{-2n} - P_{-2n-1}) + 1)$.

4

(c) $\sum_n (-1)^k P_{-2k+1} = {}^1((-1)^n (P_{-2n+4} - 3P_{-2n+3} + 3P_{-2n+1} + P_{-2n-1}) - 1)$.

4

Taking $W_n = Q_n$ with $Q_0 = 6$, $Q_1 = 2$, $Q_2 = 6$, $Q_3 = 17$, $Q_4 = 46$, $Q_5 = 122$ in the last Proposition, we have the following Corollary which presents sum formulas of sixth-order Pell-Lucas numbers.

(a)

2

Corollary 5.11. *For $n \geq 1$, sixth-order Pell-Lucas numbers have the following properties:*

n $k=1$

$k=1$

(b) $\sum_n$

$k=1$

$(-1)^k Q_{-k} = {}^1 ((-1)^n (-Q_{-n+5} + 3Q_{-n+4} - 2Q_{-n+3} + 3Q_{-n+2} - 2Q_{-n+1} + 3Q_{-n}) - 14)$. $(-1)^k Q_{-2k} = {}^1((-1)^n (Q_{-2n+4} - Q_{-2n+3} - 4Q_{-2n+2} - Q_{-2n+1} + 2Q_{-2n} - Q_{-2n-1}) - 16)$.

4

(c) $\sum_n (-1)_k Q_{-2k+1} = {}_1((-1)_n (Q_{-2n+4} - 3Q_{-2n+3} + 3Q_{-2n+1} + Q_{-2n-1}))$.

4

Observe that setting $x = -1$, $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 1$, $y = 2$ (i.e. for the generalized sixth order Jacobsthal case) in Theorem 4.1 (a), makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 5.12. *If $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 2$, $y = 2$ then for $n \geq 1$ we have the following formulas:*

$2(-1)$
$(n + 3)W_{-n+2} - (-1)$
$(n + 7)W_{-n+1} + (-1)$
$(2n + 9)W_{-n} + W_5 - 3W_4 + 4W_3 -$
$k=1$
n
9
n
n

(a) $\sum_n (-1)^k W_{-k} = {}^1(-(-1)^n (n+1)W_{-n+5} + (-1)^n (2n+3)W_{-n+4} - (-1)^n (n+4)W_{-n+3} +$
$6W_2 + 7W_1 - 9W_0)$.

$k=1$
10

(b) $\sum_n (-1)_k W_{-2k} = \frac{1}{}((-1)_n (3W_{-2n+4} - 2W_{-2n+3} - 7W_{-2n+2} - 2W_{-2n+1} + 3W_{-2n} - 2W_{-2n-1}) +$
$W_5 - 4W_4 + W_3 + 6W_2 + W_1 - 4W_0)$.

$k=1$
10

(c) $\sum_n (-1)^k W_{-2k+1} = \frac{1}{}((-1)^n (W_{-2n+4} - 4W_{-2n+3} + W_{-2n+2} + 6W_{-2n+1} + W_{-2n} + 6W_{-2n-1}) -$
$3W_5 + 2W_4 + 7W_3 + 2W_2 - 3W_1 + 2W_0)$.

Proof.

- We use Theorem 4.1 (a). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 4.1 (a) then we have where

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{g_6(x)}{-(x-2)(x+1)(x+x^2+1)(-x+x^2+1)}$$

$g_6(x) = -x^{n+1}W_{-n+5} - x^{n+1}(x-1)W_{-n+4} + x^{n+1}(-x^2+x+1)W_{-n+3} + x^{n+1}(-x^3+x^2+x+1)W_{-n+2} + x^{n+1}(-x^4+x^3+x^2+x+1)W_{-n+1} + x^{n+1}(-x^5+x^4+x^3+x^2+x+1)W_{-n} + xW_5 + x(x-1)W_4 - x(-x^2+x+1)W_3 - x(-x^3+x^2+x+1)W_2 - x(-x^4+x^3+x^2+x+1)W_1 - x(-x^5+x^4+x^3+x^2+x+1)W_0$.

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) using

$$\sum_{k=1}^{n}(-1)^k W_{-k} = \left.\frac{\frac{d}{dx}(g_6(x))}{\frac{d}{dx}(-(x-2)(x+1)(x+x^2+1)(-x+x^2+1))}\right|_{x=-1}$$

$$= \frac{1}{9}(-(-1)^n(n+1)W_{-n+5} + (-1)^n(2n+3)W_{-n+4} - (-1)^n(n+4)W_{-n+3} + 2(-1)^n(n+3)W_{-n+2} - (-1)^n(n+7)W_{-n+1} + (-1)^n(2n+9)W_{-n} + W_5 - 3W_4 + 4W_3 - 6W_2 + 7W_1 - 9W_0).$$

(b) Take $x = -1, r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ in Theorem 4.1 (b).

(c) Take $x = -1, r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ in Theorem 4.1 (b). Q

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ in the last Proposition, we have the following Corollary which presents sum formulas of sixth-order Jacobsthal numbers.

Corollary 5.13. *For $n \geq 1$, sixth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k J_{-k} = \frac{1}{9}(-(-1)^n(n+1)J_{-n+5} + (-1)^n(2n+3)J_{-n+4} - (-1)^n(n+4)J_{-n+3} + 2(-1)^n(n+3)J_{-n+2} - (-1)^n(n+7)J_{-n+1} + (-1)$

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n

$(2n+9)J_{-n}+3)$.

k=1
10

(b) $\sum_{k=1}^{n} (-1)^k J_{-2k} = \frac{1}{10}((-1)^n(3J_{-2n+4} - 2J_{-2n+3} - 7J_{-2n+2} - 2J_{-2n+1} + 3J_{-2n} - 2J_{-2n-1}) + 5)$.

k=1
10

(c) $\sum_{k=1}^{n} (-1)^k J_{-2k+1} = \frac{1}{10}((-1)^n(J_{-2n+4} - 4J_{-2n+3} + J_{-2n+2} + 6J_{-2n+1} + J_{-2n} + 6J_{-2n-1}) + 5)$.

From the last Theorem, we have the following Corollary which gives sum formulas of sixth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$).

Corollary 5.14. *For $n \geq 1$, sixth order Jacobsthal-Lucas numbers have the following properties:*

2 (−1)
$(n+3)j_{-n+2} - (-1)$
$(n+7)j_{-n+1} + (-1)$
$(2n+9)j_{-n} - 21)$.
k=1
10
k=1
n
9
n
n

(a) $\sum_{k=1}^{n} (-1)^k j_{-k} = \frac{1}{9}(-(-1)^n(n+1)j_{-n+5} + (-1)^n(2n+3)j_{-n+4} - (-1)^n(n+4)j_{-n+3} +$

(b) $\sum_{k=1}^{n} (-1)^k j_{-2k} = \frac{1}{10}((-1)^n(3j_{-2n+4} - 2j_{-2n+3} - 7j_{-2n+2} - 2j_{-2n+1} + 3j_{-2n} - 2j_{-2n-1}) - 7)$.

k=1
10

(c) $\sum_{k=1}^{n} (-1)^k j_{-2k+1} = \frac{1}{10}((-1)^n(j_{-2n+4} - 4j_{-2n+3} + j_{-2n+2} + 6j_{-2n+1} + j_{-2n} + 6j_{-2n-1}) + 1)$.

Taking $j_n = K_n$ with $K_0 = 3$, $K_1 = 1$, $K_2 = 3$, $K_3 = 10$, $K_4 = 20$, $K_5 = 40$ in the last Theorem, we have the following corollary which presents sum formula of modified sixth order Jacobsthal numbers.

Corollary 5.15. *For $n \geq 1$, modified sixth order Jacobsthal numbers have the following property:*

2 (−1)
$(n+3)K_{-n+2} - (-1)$
$(n+7)K_{-n+1} + (-1)$
$(2n+9)K_{-n} - 18)$.
(b)

k=1
n
9
n
n

(a) $\sum_{k=1}^{n} (-1)^k K_{-k} = \frac{1}{9}(-(-1)^n(n+1)K_{-n+5} + (-1)^n(2n+3)K_{-n+4} - (-1)^n(n+4)K_{-n+3} +$

n
k=1

(c)

10

23).

$(-1)^k K_{-2k} = \frac{1}{10}((-1)^n(3K_{-2n+4} - 2K_{-2n+3} - 7K_{-2n+2} - 2K_{-2n+1} + 3K_{-2n} - 2K_{-2n-1}) -$

10

n k=1
1).

$(-1)^k K_{-2k+1} = \frac{1}{10}((-1)^n(K_{-2n+4} - 4K_{-2n+3} + K_{-2n+2} + 6K_{-2n+1} + K_{-2n} + 6K_{-2n-1}) -$

From the last Theorem, we have the following corollary which gives sum formula of sixth-order Jacobsthal Perrin numbers (take $K_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$).

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Corollary 5.16. *For $n \geq 1$, sixth-order Jacobsthal Perrin numbers have the following property:*

$2(-1)$

$(n+3)Q_{-n+2} - (-1)$

$(n+7)Q_{-n+1} + (-1)$

$(2n+9)Q_{-n} - 23)$.

(b)

$k=1$

n

9

n

n

(a) $\sum_{n} (-1)^k Q_{-k} = {}^1(-(-1)^n (n+1)Q_{-n+5} + (-1)^n (2n+3)Q_{-n+4} - (-1)^n (n+4)Q_{-n+3} +$

n

$k=1$

(c)

10

24).

$(-1)^k Q_{-2k} = \frac{1}{}((-1)^n (3Q_{-2n+4} - 2Q_{-2n+3} - 7Q_{-2n+2} - 2Q_{-2n+1} + 3Q_{-2n} - 2Q_{-2n-1}) -$

10

$n\ k=1$

2).

$(-1)^k Q_{-2k+1} = \frac{1}{}((-1)^n (Q_{-2n+4} - 4Q_{-2n+3} + Q_{-2n+2} + 6Q_{-2n+1} + Q_{-2n} + 6Q_{-2n-1}) +$

Taking $Q_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ in the Theorem, we have the following corollary which presents sum formula of adjusted sixth-order Jacobsthal numbers.

Corollary 5.17. *For $n \geq 1$, adjusted sixth-order Jacobsthal numbers have the following property:*

$2(-1)$

$(n+3)S_{-n+2} - (-1)$

$(n+7)S_{-n+1} + (-1)$

$(2n+9)S_{-n} + 5)$.

$k=1$

10

$k=1$

n

9

n

n

(a) $\sum_{n} (-1)^k S_{-k} = {}^1(-(-1)^n (n+1)S_{-n+5} + (-1)^n (2n+3)S_{-n+4} - (-1)^n (n+4)S_{-n+3} +$

(b) $\sum_{n} (-1)^k S_{-2k} = \frac{1}{}((-1)^n (3S_{-2n+4} - 2S_{-2n+3} - 7S_{-2n+2} - 2S_{-2n+1} + 3S_{-2n} - 2S_{-2n-1}) + 1)$.

$k=1$

(c) $\sum_{n} (-1)^k S_{-2k+1} = \frac{1}{}((-1)^n (S_{-2n+4} - 4S_{-2n+3} + S_{-2n+2} + 6S_{-2n+1} + S_{-2n} + 6S_{-2n-1}) - 3)$.

10

From the last Theorem, we have the following corollary which gives sum formula of modified sixth- order Jacobsthal-Lucas numbers (take $S_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$).

Corollary 5.18. *For $n \geq 1$, modified sixth-order Jacobsthal-Lucas numbers have the following property:*

$k=1$

n

9

n

(a) $\sum_{n} (-1)^k R_{-k} = {}^1(-(-1)^n (n+1)R_{-n+5} + (-1)^n (2n+3)R_{-n+4} - (-1)^n (n+4)R_{-n+3} +$

$2(-1)$

$(n+3)R_{-n+2} - (-1)$

$(n+7)R_{-n+1} + (-1)$

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$(2n+9)R_{-n} - 51)$.

(b)

10

n
$k=1$
27).

$(-1)^k R_{-2k} = \frac{1}{} ((-1)^n (3R_{-2n+4} - 2R_{-2n+3} - 7R_{-2n+2} - 2R_{-2n+1} + 3R_{-2n} - 2R_{-2n-1}) -$

$k=1$
10

(c) $\sum_n (-1)^k R_{-2k+1} = \frac{1}{} ((-1)^n (R_{-2n+4} - 4R_{-2n+3} + R_{-2n+2} + 6R_{-2n+1} + R_{-2n} + 6R_{-2n-1}) + 1)$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ in Theorem 4.1 (a), (b) and (c), we obtain thefollowing proposition.

Proposition 5.3. *If $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ then for $n \geq 1$ we have the followingformulas:*

$k=1$
4

(a) $\sum_n (-1)^k W_{-k} = {}^1((-1)^n (W_{-n+5} - 3W_{-n+4} - 5W_{-n+2} - 2W_{-n+1} - 9W_{-n}) - W_5 + 3W_4 +$
$5W_2 + 2W_1 + 9W_0)$.

$k=1$
82

(b) $\sum_n (-1)^k W_{-2k} = \frac{1}{} ((-1)^n (5W_{-2n+4} - 6W_{-2n+3} - 28W_{-2n+2} - 31W_{-2n+1} - 27W_{-2n} -$

$52W_{-2n-1}) + 4W_5 - 13W_4 - 6W_3 + 8W_2 + 3W_1 - 17W_0)$.

$k=1$
82

(c) $\sum_n (-1)^k W_{-2k+1} = \frac{1}{} ((-1)^n (4W_{-2n+4} - 13W_{-2n+3} - 6W_{-2n+2} + 8W_{-2n+1} + 3W_{-2n} +$
$65W_{-2n-1}) - 5W_5 + 6W_4 + 28W_3 + 31W_2 + 27W_1 + 52W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of 6-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$).

Corollary 5.19. *For $n \geq 1$, 6-primes numbers have the following properties:*

$k=1$
4

(a) $\sum_n (-1)^k G_{-k} = {}^1((-1)^n (G_{-n+5} - 3G_{-n+4} - 5G_{-n+2} - 2G_{-n+1} - 9G_{-n}) + 1)$.

$k=1$
82

(b) $\sum_n (-1)^k G_{-2k} = \frac{1}{} ((-1)^n (5G_{-2n+4} - 6G_{-2n+3} - 28G_{-2n+2} - 31G_{-2n+1} - 27G_{-2n} -$

(c)

$52G_{-2n-1}) - 5)$.

82

n $k=1$
4).

$(-1)^k G_{-2k+1} = \frac{1}{} ((-1)^n (4G_{-2n+4} - 13G_{-2n+3} - 6G_{-2n+2} + 8G_{-2n+1} + 3G_{-2n} + 65G_{-2n-1}) -$

Taking $G_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 6-primes numbers.

(a)

Corollary 5.20. *For $n \geq 1$, Lucas 6-primes numbers have the following properties:*

n k=1

k=1

52H_{-2n-1}) − 44).

$\sum_n$

$(-1)^k H_{-k}$ = 1((−1)^n (H_{-n+5} − 3H_{-n+4} − 5H_{-n+2} − 2H_{-n+1} − 9H_{-n}) + 16).

4

82

$(-1)^k H_{-2k}$ = 1 ((−1)^n (5H_{-2n+4} − 6H_{-2n+3} − 28H_{-2n+2} − 31H_{-2n+1} − 27H_{-2n} −

k=1

82

(c) $\sum_n$ $(-1)^k H_{-2k+1}$ = 1 ((−1)^n (4H_{-2n+4} − 13H_{-2n+3} − 6H_{-2n+2} + 8H_{-2n+1} + 3H_{-2n} + 65H_{-2n-1}) + 14).

From the last proposition, we have the following corollary which gives sum formulas of modified 6-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$).

(a)

Corollary 5.21. *For n ≥ 1, modified 6-primes numbers have the following properties:*

n k=1

k=1

9).

(b) $\sum_n$

$(-1)^k E_{-k}$ = 1((−1)^n (E_{-n+5} − 3E_{-n+4} − 5E_{-n+2} − 2E_{-n+1} − 9E_{-n}) + 2).

4

82

$(-1)^k E_{-2k}$ = 1 ((−1)^n (5E_{-2n+4}−6E_{-2n+3}−28E_{-2n+2}−31E_{-2n+1}−27E_{-2n}−52E_{-2n-1})−

(c)

82

n k=1

1).

$(-1)^k E_{-2k+1}$ = 1 ((−1)^n (4E_{-2n+4}−13E_{-2n+3}−6E_{-2n+2}+8E_{-2n+1}+3E_{-2n}+65E_{-2n-1})+

• The case $x = i$

In this subsection, we consider the special case $x = i$.

Taking $r = s = t = u = v = y = 1$ in Theorem 4.1, we obtain the following proposition.

Proposition 5.4. *If $r = s = t = u = v = y = 1$ then for $n \geq 1$ we have the following formulas:*

k=1

2+i

(a) $\sum_n$ $i^k W_{-k}$ = 1 (i^n(−iW_{-n+5} + (1 + i)W_{-n+4} − (1 − 2i)W_{-n+3} − 2W_{-n+2} − iW_{-n+1} + (1 + i)W_{-n}) + iW_5 − (1 + i)W_4 + (1 − 2i)W_3 + 2W_2 + iW_1 − (1 + i)W_0).

k=1

−4+i

(b) $\sum_n$ $i^k W_{-2k}$ = 1 (i^n(2W_{-2n+4} − (2 + i)W_{-2n+3} − (2 − 3i)W_{-2n+2} − (1 + i)W_{-2n+1} − (5 +

$i)W_{-2n} - W_{-2n-1}) + W_5 - 3W_4 + (1+i)W_3 + (1-3i)W_2 + iW_1 + (4+i)W_0).$

(c) $\sum_{k=1}^{n} i^k W_{-2k+1} = \frac{1}{-4+i}(i^n(-iW_{-2n+4} + 3iW_{-2n+3} + (1-i)W_{-2n+2} - (3+i)W_{-2n+1} + W_{-2n} + 2W_{-2n-1}) - 2W_5 + (2+i)W_4 + (2-3i)W_3 + (1+i)W_2 + (5+i)W_1 + W_0).$

From the above Proposition, we have the following Corollary which gives sum formulas of Hexanacci numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$).

Corollary 5.22. *For $n \geq 1$, Hexanacci numbers have the following properties:*

(a) $\sum_{k=1}^{n} i^k H_{-k} = \frac{1}{2+i}(i^n(-iH_{-n+5} + (1+i)H_{-n+4} - (1-2i)H_{-n+3} - 2H_{-n+2} - iH_{-n+1} + (1+i)H_{-n}) + i).$

(b) $\sum_{k=1}^{n} i^k H_{-2k} = \frac{1}{-4+i}(i^n(2H_{-2n+4} - (2+i)H_{-2n+3} - (2-3i)H_{-2n+2} - (1+i)H_{-2n+1} - (5+i)H_{-2n} - H_{-2n-1}) - 1).$

(c) $\sum_{k=1}^{n} i^k H_{-2k+1} = \frac{1}{-4+i}(i^n(-iH_{-2n+4} + 3iH_{-2n+3} + (1-i)H_{-2n+2} - (3+i)H_{-2n+1} + H_{-2n} + 2H_{-2n-1}) + 2).$

Taking $H_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$ in the above Proposition, we have the following Corollary which presents sum formulas of Hexanacci-Lucas numbers.

Corollary 5.23. *For $n \geq 1$, Hexanacci-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} i^k E_{-k} = \frac{1}{2+i}(i^n(-iE_{-n+5} + (1+i)E_{-n+4} - (1-2i)E_{-n+3} - 2E_{-n+2} - iE_{-n+1} + (1+i)E_{-n}) + (-8-3i)).$

(b) $\sum_{k=1}^{n} i^k E_{-2k} = \frac{1}{-4+i}(i^n(2E_{-2n+4} - (2+i)E_{-2n+3} - (2-3i)E_{-2n+2} - (1+i)E_{-2n+1} - (5+i)E_{-2n} - E_{-2n-1}) + (20+5i)).$

(c) $\sum_{k=1}^{n} i^k E_{-2k+1} = \frac{1}{-4+i}(i^n(-iE_{-2n+4} + 3iE_{-2n+3} + (1-i)E_{-2n+2} - (3+i)E_{-2n+1} + E_{-2n} + 2E_{-2n-1}) + (-4-2i)).$

Corresponding sums of the other sixth order generalized Hexanacci numbers can be calculated similarly.

• Conclusion

Numerous studies of numerical sequences have been published in the literature recently, and these sequences have been extensively employed in a wide range of scientific fields, including physics, engineering, architecture, nature, and the arts. It was demonstrated in this work that linear sum identities exist. The technique employed in this research is also applicable to the other linear recurrence sequences. The linear sum identities have been written in terms of the generalised Hexanacci sequence. The corresponding identities for the Hexanacci, Hexanacci-Lucas, sixth order Pell, sixth order Pell-Lucas, sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth-order Jacobsthal Perrin, adjusted sixth-order Jacobsthal, modified sixth-order Jacobsthal-Lucas, 6-primes, Lucas 6-primes, and modified 6-primes sequences have been presented as special cases. Induction can be used to prove each of the identities stated in the corollaries, but it provides no information regarding how those identities were found. We provide the evidence to show how these IDs were generally found.

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